SMITHSONIAN MISCELLANEOUS COLLECTIONS.

— 843 —

# THE MECHANICS

OF THE

# EARTH'S ATMOSPHERE.

A COLLECTION OF TRANSLATIONS

BY

CLEVELAND ABBE.

CITY OF WASHINGTON:
PUBLISHED BY THE SMITHSONIAN INSTITUTION.
1891.

# CONTENTS.

# THE MECHANICS OF THE EARTH'S ATMOSPHERE:

## A COLLECTION OF TRANSLATIONS.

By Cleveland Abbe.

### INTRODUCTION.

The complexity of the phenomena of the atmosphere has rendered it necessary to delay their mathematical treatment until our knowledge of hydro-dynamics and thermo-dynamics could attain the perfection which it began to acquire about the middle of this present century at the hands of Helmholtz, Clausius, Sir William Thomson, and their disciples During the past few years some of the fundamental problems of meteorology have been treated analytically and graphically with great success. The present collection of translations presents some of the best memoirs that have lately been published on the respective subjects by European investigators; a few earlier memoirs of great excellence are included in the collection because of the references subsequently made to them. Other mathematical memoirs by Guldberg and Mohn, Marchi and Diro Kitao have been omitted because their length would have made this collection too large for the present mode of publication.

There is a crying need for more profound researches into the mechanics of the atmosphere, and believing as I do that meteorology can only be advanced beyond its present stage by the devotion to it of the highest talent in mathematical and experimental physics, I earnestly commend these memoirs to such students in our universities as are seeking new fields of applied science.

I have taken a very few liberties in translating the language and notation of the distinguished authors whose works are here collected. I have frequently used the word *liquid* instead of "Wasser," "Tropfbar-Flussigkeit," "Inkompressible Flussigkeit," and the word *gas* or *vapor* as equivalent to compressible or elastic fluid, and have used the word *fluid* when the more general term including liquids, vapors, and gases is needed. As the ideal or "perfect" liquid is absolutely incompressible and devoid of all resistance to mere change of shape, having neither elasticity nor viscosity, namely, internal friction, it seems more proper

to use the general terms liquid, gas, and fluid when neglecting the resistance, compressibility, elasticity, and viscosity as in dealing with these ideal substances, and to reserve the terms air, water, etc., for use when dealing with actual natural fluid phenomena where slight compressions and expansions and resistances occur.

The relation between elastic pressure, volume, and temperature, as deduced by Boyle, Mariotte, Gay-Lussac, and Charles, that characterizes a gas, and the equation for which the Germans call the "Zustands-Gleichung" in common with other equations of condition, I have preferred to speak of as the *equation of elasticity* or the characteristic equation of a perfect gas.

In view of the remarkable want of uniformity existing in English and American works in respect to the notation for total and partial differentials I have decided to make such alterations in the original notations of these papers as shall make the whole series consistent with the elegant and classical notation that is rapidly being adopted in Germany, and that will, I hope, eventually be accepted by all English and French writers. In accordance with this I shall always express the *total differential* by $d$, as first introduced into geometry by Leibnitz for the infinitesimal difference; the small increment or *variation* by $\delta$, as introduced by Lagrange; the large *finite difference* by $\Delta$, first used by Euler; the *partial differential* by $\partial$, ("the round d,") as used by Jacobi. Occasionally the dotted variable $\dot{x}$ will indicate the rate of variation with regard to the time, or the fluxion as first introduced into mathematical physics by Sir Isaac Newton, a notation which has lately been extensively revived in England by those devoted to classic authority.

Evidently the problems here treated by elegant mathematical methods are not always precisely the problems of nature. The differences between the conclusions of Rayleigh, Margules, and Ferrel as to the diurnal and semi-diurnal tides due to heat, or the differences between Ferrel, Oberbeck, and Siemens on the one hand and nature on the other as to the general circulation, show that by the omission of apparently minor local and periodical irregularities we have constructed for ourselves problems that still differ from the case of the earth's atmosphere, although they may more closely represent the conditions of such a planet as Jupiter.

I have to acknowledge the assistance of my friend, Mr. G. E. Curtis, in copying a portion of the formulæ for these translations, and renew the expression of my hope that a coming generation of American meteorologists may prosecute to further conquests the mathematical studies begun by Ferrel and perfected by our European colleagues.

CLEVELAND ABBE.

FEBRUARY, 1891.

## THE MEASUREMENT OF THE RESISTANCES EXPERIENCED BY PLANE PLATES WHEN THEY ARE MOVED THROUGH THE AIR IN A DIRECTION NORMAL TO THEIR PLANES.*

By Professor G. H. L. HAGEN.

Some time since I submitted to the Academy the results of a series of observations that I had instituted upon the motions of air and of water when the uniform flow of these fluids is interrupted by means of interposed planes.† By means of small bits of paper or tin foil floating from the tips of needles the direction of the motion could be perceived at every point. The velocities were indeed too feeble to be capable of direct measurement, but the disposition of particles of pulverized amber that were strewn over the water showed the limits of the strongest current, and when the coarser particles came to rest before the finer ones it was to be inferred that there was a gradual diminution of velocity at such points.

In general it was concluded that air and water alike swerve in curved paths in front of such obstacles and flow towards the free openings. In the latter and directly adjoining the outer ends of the obstacle the strongest current is formed which here retains its direction unaltered, therefore free from all variations. The deviation in front of the obstacle does not take place at any definite distance from it, but rather extends up to the obstacle itself and even when the plate faces the current it is seen that a feeble motion still exists immediately adjoining it.

Behind the obstacle the fluid by no means remains at rest, but rather there is always formed here a counter current whose length is equal to four or five times the distance of the head of the obstacle from the neighboring side wall of the channel, which counter current, however, is not only fed at its rear end, but principally also at two intermediate points by the steadily broadening main current. The latter immediately behind the head of the cross-wall meets the outcoming counter-current

*Read before the Academy of Sciences, Berlin, January 22, February 16, and April 20, 1874. (Translated from The Mathematical Memoirs [Abhandlungen] of the Royal Academy of Sciences at Berlin for the year 1874, pp. 1 to 31.)

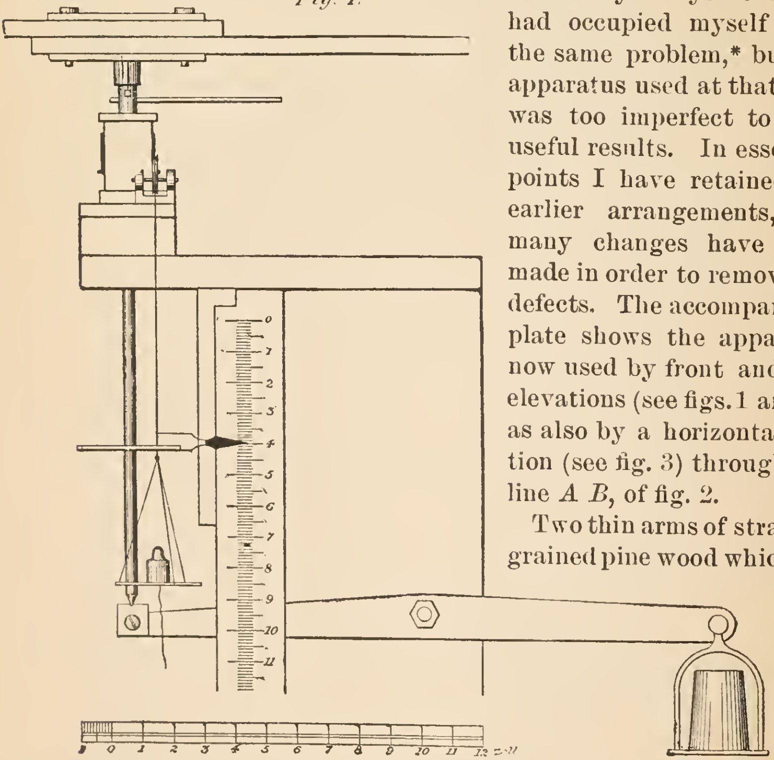
†See the Monats-Berichte for 1872, p. 861.

and here, as also at the two intermediate places just mentioned, whirls are formed which set in rotation the little vanes placed there. The phenomena agree with those that one observes in streams and rivers in front of and behind sharp protruding rocks or piers.

It must still be mentioned that neither water nor air rebounds like elastic spheres from the obstacle against which it strikes, as is frequently assumed. Even strong streams of water that I allowed to play against the plates did not rebound, but continued their onward path close to the obstacle, producing a strong current there.

I had instituted these experiments in order to see in what manner the resistances originate that the liquid experiences in such deviations and which cause the pressure against the opposing plate. However, I thought it was allowable to assume that when the plate is itself moved through stationary water or air the ratios remain nearly the same and that similar currents of the fluid occur in its neighborhood. The pressure that the plate experiences in this latter case is the object of the following investigation which is moreover confined to plane disks moved through the air in a direction perpendicular to their planes.

*Fig. 1.*




Already 40 years ago I had occupied myself with the same problem,* but the apparatus used at that time was too imperfect to give useful results. In essential points I have retained the earlier arrangements, but many changes have been made in order to remove the defects. The accompanying plate shows the apparatus now used by front and side elevations (see figs. 1 and 2), as also by a horizontal section (see fig. 3) through the line $A$ $B$, of fig. 2.

Two thin arms of straight-grained pine wood which are

*Some of the series of observations made at that time are communicated as examples of the application of the method of least squares in the first edition of the "Grundzuge der Wahrscheinlichkeits-Rechnung."

bevelled on the sides that cut through the air, rest upon a vertical metal axis which communicates the rotary motion to them. Each of these arms is 8 feet or 96 Rhenish inches long and on its end the disk is fastened whose resistance is to be measured. In order to prevent the bending of the arms they are held not far from their ends by small wires which pass over a support 18 inches high vertically above the vertical axis. The drawing presents only the connection of the two arms between themselves and with the axis. The latter is in its upper portion turned slightly conical and carries the corresponding hollow hub which is screwed to the brass plate under the arms.

*Fig. 2.*

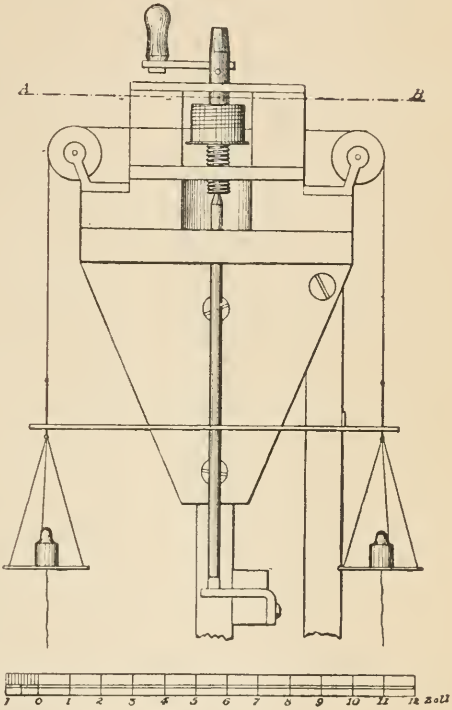
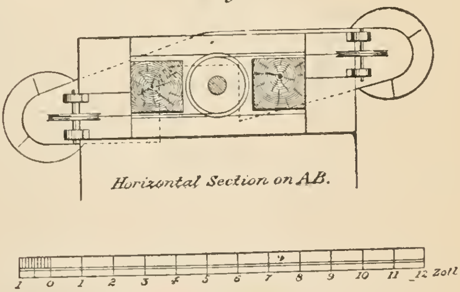



*Fig. 3.*




*Horizontal Section on AB.*

The rotation is brought about by the tension of two small threads which are wound in the same direction around the ivory spindle that is fastened to the axis, and are then drawn in opposite directions over two rollers and drawn taut by light scale pans with weights therein. These rollers I had formerly fastened at the greatest possible distance on the opposite walls of the room in order that when winding up the weights the threads might lie uniformly alongside of and not over each other, but this design was by no means certainly attained and the far-stretched threads materially increased the labor of the observation, especially since the arms and the disks fastened to them occasionally came in contact with these threads.

When in the past summer I again undertook the observations I placed the rollers, as the drawing shows, close to the axis, but did not let the latter stand upon a fixed point, but rather provided it with a screw thread on its lower part whose mother is cut into a thick plate of brass. By rotation the axis therefore rose or sank uniformly,

and the threads simultaneously arranged themselves alongside of each other on the spindle from which they were drawn always in a horizontal direction. Underneath the roller I connected both threads by means of a light rod, and on this hung the scale-pans for weights; I also fastened thereon a pointer which slid close to the graduated scale and served to measure the velocity.

Notwithstanding the great convenience of this change it introduced the troublesome consideration that the friction became disproportionately great and varied so much during the observation that its magnitude and its influence on the measured velocity could not be determined with the necessary accuracy. This great evil I removed in that I allowed a steel point to work in the conical depression already formed by the turning lathe at the lower end of the axis, which point exerted an upward pressure equal to the weight of the arms, the discs, and the axis. The axis is therefore completely supported by the steel point, and the screw serves only as a guide in order to raise and lower the spindle corresponding to the windings of the thread. This steel point forms the upper end of a stout wire 12 inches high, whose lower end, ground to a wedge shape, stands in a metallic groove that is fastened at the end of a lever whose equal arms are 19 inches long. This lever, whose center of gravity lies in its axis of rotation, was so formed that its axis lay in a straight line with the metal groove and the point of suspension of the scale-pan, and was equally distant from both. This pan, with the counterpoise, corresponded exactly to the pressure of the axis on the wire when no resisting discs were placed upon the arms, but as soon as the latter occurred the counterpoise was always increased in a corresponding degree by an appended light cup with shot. Before attaching the discs these were laid upon a balance and the cup was partly filled with shot until brought into equilibrium with it.

Since the lever changes its position during the rotation of the axis the steel wire deviates somewhat from the vertical position, but, as will be shown in the following, so slightly that this may be overlooked. The result of these changes in the apparatus proved to be very favorable, for whereas before at least 3 Prussian loths had to be placed in each scale in order to set the arms in permanent motion, now, the weight of the rod and the scale-pan, which together weighed 3.3 loths, sufficed without any additional weight to produce a uniform motion.

At the ends of the arms pieces of perforated cork are glued, and in these the stems of the various discs find their support. The discs were always pushed so far on that they closely touched the ends of the arms. The distance of the disc from the axis of rotation is found from the known lengths of the arms; the stems of the discs did not extend through the corks, therefore the resistance of the air against the arms was only increased by that which the discs themselves experienced. Therefore, after the resistance which the arms experienced at each velocity had been determined by observation of the rotation of the

arms under various loads, this could be subtracted each time from the resistance observed with the disc in place and thereby the resistance of the various discs for various velocities be determined.

The ivory spindle around which the threads wind was, like the axis, very carefully turned cylindrical, and is 1.1 inches high and 1.6 inches in diameter. The portion of the axis extending above the spindle is also turned cylindrical so that for any position of the upper perforated brass plate it is securely held with very little play. Under its slightly conical flat head are found, as the figure shows, two openings perpendicular to each other, one square and the other circular. The first serves for the introduction of a small crank handle by means of which the axis is turned backwards when the weights are being raised. Before taking off the arms a wire is put through the circular opening which prevents the axis from turning forward while the observer is taking off the crank and putting on the arms and discs. Moreover, at a distance of 12 inches from the axis there is placed a bent lever, one arm of which stands upright and hinders the turning of the arms that carry the discs when the weight fastened to the other arm of the bent lever hangs freely. While the arm carrying the discs is thus held by the bent lever the stout wire is withdrawn and the air is allowed to come to rest. If the weight be placed on the neighboring table then the bent-lever arm falls and the apparatus starts in motion.

The pitch of the screw of the axis below the spindle is 0.05 inch, and this distance corresponds to the width of both threads so that the latter lie regularly close to each other on the surface of the spindle. This always occurred very regularly even when the axis was turned very rapidly by means of the crank handle. The threads, the so-called "iron twine," were so strong that each with safety carried 4 pounds, which weight, however, was never even distantly approached in practice. The threads were so light that 40 feet weighed only 0.1 loth, so that the fall of the index by 6 feet increased the driving power by only 0.03 loth. Nevertheless, for very feeble loads in the scale-pans a slight increase in the velocity was apparent during the descent, and in order to prevent this the small increase in the weight was annulled by means of two equal threads suspended from the scale-pans to the floor.

Since the two former or driving threads were fastened to the rod they were thereby prevented from turning and unwinding, which I had been able to avoid in my earlier work only by guiding the scale-pans by means of taut wires. Even if, however, the threads by this method of fastening did not materially change, still it remained to be proved whether perhaps they lengthened sensibly with greater tension, in which case the relation between the path of the index and the rotation of the arms could not remain constant. Such an extension could not be mistaken when I laid a weight of 1 pound on the empty scale-pans when they were at their lowest position. The index then sank at once 0.2 of an inch. A further extension, however, did not follow; at least

it was not to be observed in the short interval occupied by each separate observation. In consequence of this extension of the threads it was incumbent to lay those weights that were to be used to set the axis in motion during the next observation upon the scale-pan while the latter was at its lowest position. The threads were therefore always wound up under the same tension with which they were to do the work.

The question now arose whether with stronger tensions the spiral windings of the threads perhaps lay flatter on the spindle than with weaker tensions, and whether therefore the length of a winding or the path that the index described for one turn of the arms became shorter. This point was decided in that with various loads in the scale-pan I measured the path that the index described during a certain number of revolutions. The above-mentioned bent lever offered the opportunity of always stopping the arms at the same point, but it was necessary to bring them to rest by gentle pressure, because with a strong blow against the upright standing arm the horizontal arms carrying the discs could easily turn somewhat on the conical head of the axis. After the position of the index was read off I allowed the arms to make five complete turns and again read off the position of the index on the scale, estimating only to the hundredth part of an inch.

The lengths of the paths for the corresponding weights in each scale-pan are as follows:

| Weight.* | Path.† |
|---|---|
| 0 | 25.69 |
| 4 | .67 |
| 8 | .68 |
| 16 | .66 |
| 24 | .67 |
| 28 | .65 |

* Prussian loths. † Rhenish inches.

A very slight shortening of the path appears from this to occur for the heavier loads, but if it actually exists it is so small that it is far less than the accuracy of the measurement of the path of the index on the divided scale. It may therefore be assumed that the velocity of the index stands in a constant ratio to that of the arms or disks.

The lengths of the individual windings of the thread around the spindle as resulting from the above measures do not correspond in all accuracy to the circumference of a circle that is normal to the axis of the spindle, and at a distance therefrom equal to that of the central axis of the threads, inasmuch as the threads lie spirally around the spindle. Now the pitch of the screw measures 0.05 inch; therefore the threads on the surface of the spindle make an angle with the horizon 0° 33′ 29″. Since the average length of one winding of the thread is 5.134 inches, therefore the equivalent thread encircling the normal is somewhat smaller, namely, 5.1338. Hence the resulting distance of the center of the threads from the axis of rotation or the length of the lever arm by

which the weight acts is equal to 0.81705 inch. This figure is adopted in the following computations, where it is represented by the letter $a$.

It remains still to investigate whether the steel wire that carries the axis may perhaps depart so far from the vertical direction by the movement of the lever on which it rests that it occasionally may exert an appreciable side pressure and thereby in an injurious way increase the friction in the screw threads. The lever is, as was mentioned, not only perfectly balanced, but the point that carries the counterpoise is also situated in the prolongation of the straight line drawn through the supporting point of the wire and the rotation axis of the lever. Therefore for every position of the lever the foot of the wire is pressed upwards vertically with equal force, but it rises only 0.8 of an inch, while the weight that drives the disks around in the extreme case sinks 80 inches. Therefore the deviation of the foot of the steel wire from a mean position amounts only to 0.4 of an inch, or in angle $2^\circ\ 24'\ 48''$, for a length of the lever arm of 9.5 inches. Therefore the deviation from the initial verticality is limited to 0.0086 inch, and consequently the wire 12 inches long is inclined $0^\circ\ 2'\ 38''$ to the vertical. Even this small inclination can be reduced by one-half if we place the axis of the wire or its upper point in the vertical line that bisects the deviation of its lower end, but such accuracy in the establishment of the apparatus must not be anticipated. It is evident from this that there can be no sensible increase of the friction in consequence of the movement of the lever.

As regards the execution of the observations the remark must be prefaced that the Rhenish inch, or the twelfth part of the Prussian foot according to the earlier determination of the standard, and the old Prussian loth, of which 32 make 1 Prussian pfund, have been adopted as units of length and weight.* The divided scale over which the index glides is divided into tenths of inches, but this subdivision is only used for determining the length of one winding of the thread, as previously described. In all other cases only the transit of the index over the heavier division marks for each 10 inches was observed by the beating of the seconds clock and the corresponding whole or half seconds noted.

Since at the beginning of an observation the arms do not immediately assume that velocity for which the resistance in connection with the friction balances the acceleration, therefore the significant observations began only when the weight had fallen 20 inches or the index had passed over the twentieth inch mark. At the seventieth inch the weight-scale pan had approached the floor, and therefore here the measures must be stopped. When, however, the rotation of the arms was observed without disks and the weights employed were very slight, then the speed continued increasing somewhat longer and the time of transit over the twentieth inch could not be used in the calculations.

[* One Rhenish inch = 1.0297217 English inch = 26.15446 millimetres. One Prussian loth = 0.032226 pounds avoirdupois = 14.616 grammes. (See Barnard's Weights and Measures, C. A.]

In order to determine with the greatest accuracy the resistance of the air against each separate pair of disks it certainly would have been advantageous to employ very different weights and thereby attain very different velocities. This intention, however, could not be carried out by reason of the moderate length of the arms, which was limited by the dimensions of the room. If I loaded each scale pan with more than 1 pfund then the whole mass of air in the room, especially when using larger disks, assumed a rotatory motion, in which case the resistance during the individual observation is always less or the velocity is always greater. Even with a load of 1 pfund the light paper vanes that floated at the tips of the needles already showed a feeble continuous rotation, although the flame of a candle did not allow of its recognition. In all the following observations therefore in the extreme cases only 28 loth was placed in each scale pan. To this it is to be added also that the measurements for very large velocities lose in accuracy on account of the relative magnitude of the unavoidable error. According to this the index should not move faster than an inch in 1.8 seconds. On the other hand, however, on account of the excessive influence of the very variable friction, the movement became highly irregular, when more than 8 seconds elapsed while the index described 1 inch. Within these limits the times in which 10 inches were described did not easily deviate more than half a second from the average value. The velocities of the disks were therefore not greater than 66 and not less than 17 inches per second.*

In order to attain a uniform tension with reference to the axis the weights placed in the two scale pans were always equal and since on each occasion the disks attached to the arms were also always of equal magnitude, therefore each of these weights corresponded to the resistance of one disk. To this indeed should still be added one-half of the weight of the rod and the two scale pans but this may be disregarded since for each individual observation the value of the constant term which indicates the friction has to be especially computed. This constant term will then be the sum total of these weights less the friction, and presented itself always with the negative sign because the friction remained less than the weight of the rod and the scales.

In order to simplify the computation I have at first referred not to the velocity of the disks, but only to that of the index, whence as above mentioned the velocity of the rotation can be easily deduced. In this way the opportunity was offered at each observation with disks to take into consideration that resistance which the arms alone experienced for the corresponding velocity of rotation.

Before and after each series of observations, which generally occupied 3 or 4 hours, the barometer and thermometer were read off, the latter being at the same altitude above the floor as that at which the arms revolved. The computed coefficients of resistance were re-

* Between 3.5 miles and 0.9 mile per hour.

duced to the barometric pressure of 28 Paris inches and the temperature 12° Réaumur or 15° C. Assuming that the resistance of the air is proportional to its density I formed a table of the logarithms of this correction whereby the separate reduction is very easy. In case the temperature sensibly changes during the time of observation it must be assumed that this change occurred gradually and therefore for each individual observation the correction corresponding to the time is adopted. When especially large variations occurred readings were also made in the intervals; still, in such cases very large deviations were sometimes apparent, and it was repeatedly remarked that then the movement of the arms steadily increased or that the times in which the index sank 10 inches became smaller the lower its position was, which never occurred with uniform temperature. The reason of this is certainly nothing else but this, that the equilibrium of the warmer and colder air in the room gave rise to special currents that were combined with the movement of the disks. When the temperature during a series of observations changed by two degrees or more, the results deduced became so discrepant that they had to be rejected as entirely useless. For this reason the room before and during the observation could not be heated warm. On the contrary, the oven used for heating the room must be cooled down completely. Even when the sun shone on the window whose shutters could not hinder the warming, nothing remained but to stop the observations.

Almost equally troublesome was the friction in the various parts of the apparatus. This varied perpetually, wherefore its value for each individual observation had to be especially determined. Of course it diminished when fresh oil was introduced between the rubbing surfaces, but then the variations became of such magnitude and were often so sudden that the observations were again useless. Only after many days and after the arms had remained for a long time continuously in motion there was established a greater regularity. When this, however, became evident from the measures immediately following each other, then again on the next day the conditions would be remarkably changed. It was therefore necessary that the whole of any series of observations that were to be compared among themselves should be made in immediate succession. In order to render this possible it was necessary to reduce the number of measures as much as was any way allowable, namely, to the number of the desired constants. Such a course is defensible also because the individual readings, in a long series of observations, accord much more closely with the law deduced therefrom than with the similar measures repeated at other times.

These preliminary remarks are the result of the great number of observations that I have executed during a half year. These were, especially at the first, extremely unreliable, and only gradually were all the circumstances perceived that come into consideration. The following observations, which are the only ones serving as a basis for the sub-

sequent computations, were made at recent dates with the greatest possible care and under quite favorable external conditions.

The resistance that the arms alone experience for different velocities must first be determined because this must be subtracted every time from the total resistance of the disc and the arms. The following table contains the measures made on this point. $G$ is the weight [in loths] that is placed in each scale-pan, and $t$ the number of seconds occupied by the index in passing over 1 inch. The velocity of the index is therefore equal to $\frac{1}{t}$ according to the adopted unit of measure. The observations were made twice for each load in the scale-pan, and in the second column of the table the two values $t_1$ and $t_2$ thus found are given separately, while the third column contains the mean value ($t$) adopted in the succeeding computation.

| $G$. | $t_1$. | $t_2$. | $t$. | $A$. | Diff. | $B$. | Diff. |
|---|---|---|---|---|---|---|---|
| 0.0 | 5.725 | 5.725 | 5.725 | 0.040 | +0.040 | —0.009 | —0.009 |
| 0.5 | 4.238 | 4.225 | 4.2315 | 0.514 | +0.014 | +0.498 | —0.002 |
| 1.0 | 3.488 | 3.500 | 3.494 | 1.001 | +0.001 | 1.007 | +0.007 |
| 2.0 | 2.725 | 2.735 | 2.730 | 1.979 | —0.021 | 2.006 | + 6 |
| 3.0 | 2.300 | 2.312 | 2.306 | 2.986 | —0.014 | 3.018 | + 18 |
| 4.0 | 2.038 | 2.038 | 2.038 | 3 972 | —0.028 | 4.001 | + 1 |
| 6.0 | 1.700 | 1.700 | 1 700 | 5.941 | —0.059 | 5.946 | — 054 |
| 8.0 | 1.475 | 1.675 | 1.475 | 8.066 | +0.066 | 8, 029 | + .029 |

Earlier observations had shown that the resistances could be expressed by the simple formula

$$G = z + \frac{1}{t^2} S$$

On attempting to introduce a third term containing as factor the first power of the velocity the constant coefficient corresponding had a very slight value and even sometimes a negative one. Therefore I now first chose the preceding expression, and by the method of least squares found

$$z = - \ 0.531$$
$$S = + 18.703$$

By the introduction of these constants I obtained the values for $G$, which are given in the column headed $A$. The next following column shows the error or the differences ($A—G$) for each of the weights actu ally used. We notice that these errors progress very regularly in that both for the smallest and largest values of $G$ they attain the largest positive values while between these they become negative. From this circumstance it may be inferred that the form of the formula has not been appropriately chosen, and I therefore repeated the computation using the expression

$$G = z + \frac{1}{t} p + \frac{1}{t^2} s$$

This then gave,

$$z = -\ 0.724$$
$$p = +\ 1.034$$
$$s = +15.518$$

According to this last we obtain for $G$ the values given in the column headed $B$, whose errors $B - G$ are shown in the last column. We remark that these latter do not occur regularly, owing to the change of the signs for the heavier weights, and therefore can be looked upon as accidental errors of observation. The sum of the squares of the errors amounts in the last case to 0.004252, whereas in the first case it was 0.011055, therefore more than twice as great.

There is still another reason that favors the introduction of the first power of the velocity. So long as I neglected this term there occurred without exception the inexplicable phenomenon that for observations with disks the numerical value of the constant $z$ after the negative sign was always greater, therefore the friction was always smaller, the larger and heavier the disks were. This anomaly disappeared upon the introduction of such a second term.

There is, moreover, as the observations show, a peculiar condition in connection with the second term. The coefficient $p$ assumes a very small value or entirely disappears when the screw on the axis is freshly oiled. From this we may conclude something as to its significance, *i. e.*, it indicates the resistance that arises from the viscosity of the oil and which is proportional to the velocity.

When disks are attached, the resistance peculiar to them is found when we subtract from the observed resistance that which the arms experience for equal velocities. This latter, however, is so variable that we must measure it anew every time, and since it assumes various values within even short intervals, therefore there remains only one method to determine the value of the three constants $z$, $p$, and $s$, namely, to allow the arms to revolve alone with three different velocities both before and after each observation. When, however, as usually happened, a second measure again gave somewhat different values, then the appropriate mean value corresponding to the intervening time should be used in the computation.

In the resistances of the disks found in this manner the second term proportional to the velocity is no longer contained, because the influence of the viscosity of the oil has already been allowed for in the resistances of the arms. The constant $z$ is, on the other hand, so variable that it must be specially deduced from each series of observations.

The following observations were made with two square disks of 6 inches on each side.* $G'$ indicates the weight placed in each scale pan, and this changes to $G$ when we subtract the weight required to overcome the resistance of the arms for equal velocities. The second column contains as before the times during which the index sinks by 1 inch, as found from the two measurements respectively.

| $G'$ | $t$ | $t$ | $t$ | $G$ | $A$ | Diff. |
|---|---|---|---|---|---|---|
| loth. | sec. | sec. | sec. | loth. | loth. | loth. |
| 1 | 9.42 | ........ | 9.42 | 1.117 | 1.064 | −0.053 |
| 2 | 7.32 | ........ | 7.32 | 1.986 | 1.983 | − .003 |
| 3 | 6.22 | ........ | 6.22 | 2.860 | 2.875 | + .015 |
| 4 | 5.51 | 5.54 | 5.525 | 3.739 | 3.734 | .005 |
| 6 | 4.62 | 4.63 | 4.625 | 5.492 | 5.472 | − .020 |
| 8 | 4.02 | 4.04 | 4.03 | 7.231 | 7.314 | + .083 |
| 12 | 3.35 | 3.33 | 3.34 | 10.737 | 10.798 | − .061 |
| 16 | 2.92 | 2.92 | 2.92 | 14.251 | 14.235 | − .016 |
| 20 | 2.62 | 2.64 | 2.63 | 17.770 | 17.625 | − .145 |
| 24 | 2.39 | 2.40 | 2.395 | 21.247 | 21.323 | − .076 |
| 28 | 2.23 | 2.22 | 2.225 | 24.760 | 24.760 | .000 |

Adopting the expression,

$$G = z + \frac{1}{t^2} r$$

I find as most probable values

$$z = - \quad 0.335$$
$$r = + 124.24$$

From this the values of $G$ given in the column marked $A$ are deduced for the respective times. The errors of these, as contained in the following column, vary so much in sign that we can consider them as accidental and there is no reason to introduce still another term in the above expression. In this connection it must still be mentioned that when in the computation of the earlier observations I have assumed the coefficient $p$ equal to zero, a satisfactory agreement of the resistances appears for larger disks as soon as I set the resistance proportional to the square of the velocity. This is explained by the fact that the value of the term $\frac{p}{t}$ is very small in comparison with the stronger resistances which the disks experience.

[*In all that follows it is to be understood that before and after each series for the determination of the resistance of the disks a special series has been made with the arms without disks for the determination of the combined correction for arms plus friction, and that thence the correction for the resistance due to the arms has been computed. $G'$ is the weight required to overcome the friction plus the resistance of the disks and arms; $G$ is the weight required to overcome the friction plus resistance of the air to the motion of the disks; $z$ is the weight required to overcome the friction; $\frac{r}{t^2}$ is the weight required to overcome the resistance to the disks.—C. A.]

The resistance of the air against the disks is therefore proportional to the square of the velocity, and a single observation would suffice to give the coefficient $r$ if the value of $z$ were known, but since this is so very variable, therefore at least two observations at two different velocities are necessary. The further extension of the measures is unnecessary, as already before mentioned, because the greater accuracy attained surpasses the other inevitable errors; but for greater security and especially to avoid possible mistakes I have always repeated these two measures, and in such a way that beginning with the less velocity I then execute the two measures with the greater velocity and finally return again to the less.

From the values of $r$ found in this manner the pressure that the disk experiences for various velocities is directly given. Let $a$ be the known distance of the axis of rotation from the center of the threads wound round the spindle and $R$ the distance of the same axis from the center of pressure of the air against the disk, then this pressure becomes

$$D=\frac{a}{R}(G-z)=\frac{a}{t^2 R}r$$

But $\frac{1}{t}$ is the velocity of the thread, hence the velocity of the center of pressure of the disk is

$$c=\frac{R}{at}$$

and

$$D=\frac{a^3}{R^3}r\,c^2.$$

if we introduce the pressure on a unit of surface, since $F$ is the whole surface of the disk, we have

$$\frac{D}{F}=\frac{a^3}{R^3}\cdot\frac{r}{F}c^2=kc^2$$

In order to reduce the constant $r$ to the barometric pressure of 28 inches or 336 Paris lines, and to reduce the temperature to 15° C., we have for an observed pressure, $\lambda$, in Paris lines, and an observed temperature $\tau$ in centigrade degrees during the observations the reduced $r$

$$=\frac{336}{\lambda}(0.9480+0.00347\tau)\,r.$$

The distances $R$, on account of the great lengths of the arms in comparison with the width of the disks, agree quite nearly with the distances of their centers of gravity from the axis of rotation, but they are always somewhat larger and there is no reason to omit this correction, which is easily executed.

We consider first a rectangular disk whose height is $h$ and width $b$. As the origin of abscissas we may take its center of gravity whose distance from the axis of rotation is $A$, and consider the disk divided into

elementary portions, the area of any one of which is $hdx$, and the pressure that it experiences is

$$dD=\frac{Kh}{a^2t^2}(A+x)^2\,dx$$

consequently the pressure against the whole disk, found by taking the integral from $x=-\frac{1}{2}b$ to $x=+\frac{1}{2}b$, is

$$D=\frac{Khb}{a^2t^2}(A^2+\tfrac{1}{12}b^2)$$

or the average normal pressure on a unit of surface is

$$\frac{D}{F}=\frac{K}{a^2t^2}(A^2+\tfrac{1}{12}b^2).$$

If now I seek that value of $x$ which belongs to the elementary area that experiences a pressure the same as this average, then it represents the center of pressure for the whole disk. The result is,

$$A+x=R=\sqrt{A^2+\tfrac{1}{12}b^2}$$

For circular disks we again take $A$ as the distance of the center from the axis of rotation, while the radius of the disk is $\rho$. In the division of the disk into elementary vertical sections I indicate the limits of these by the angle $\varphi$ which is measured from the horizontal diameter. The area of such a section is then

$$2\,\varphi\sin\varphi^2\,d\,\varphi$$

and the pressure that it experiences is

$$dD=\frac{2\,K\rho^2}{t^2}\left(\frac{A+\rho\cos\varphi}{a}\right)^2\sin\varphi^2\,d\varphi$$

By expanding the binomial and converting the $\cos^2\varphi$ and $\cos^4\varphi$ into the sines of the multiple angle, the integration becomes very simple and the greater part of the terms disappear since the integral is taken from $\cos\varphi=-1$ to $\cos\varphi=+1$. We obtain

$$D=\frac{K\rho^2}{t^2a^2}(A^2+\tfrac{1}{4}\rho^2)\,\pi$$

or

$$\frac{D}{F}=\frac{K}{t^2a^2}(A^2+\tfrac{1}{4}\rho^2)$$

and the section that experiences this average pressure is that whose $\varphi$ satisfies the equation—

$$A+\rho\cos\varphi=R=\sqrt{A^2+\frac{1}{4}\rho^2}$$

It follows that in both kinds of disks the difference between $A$ and the desired $R$ remains very small when, as in my apparatus, $A$ is very large compared with $b$ and $\rho$.

Next, a series of observations will be communicated, made with five pairs of circular disks, whose diameters were 2.5, 3.5, 4.5, 5.5, 6.5 inches. Each time only two different weights were laid in the scale-pan; with

these, however, as above mentioned, the measures were executed twice. The resulting values of $z$ and $r$ are given in the last columns. The other letters correspond to those above given:

| $\rho$ | $G^1$ | $t_1$ | $t_2$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|
| 1.25 | 0.75 | 5.42 | 5.42 | 5.42 | —0.041 | | |
| | 9.0 | 2.00 | 1.988 | 1.994 | +4.058 | —0.683 | 18.850 |
| 1.75 | 1.5 | 5.31 | 5.30 | 5.305 | 0.690 | | |
| | 14 | 2.00 | 1.98 | 1.990 | 9.054 | —0.679 | 38.545 |
| 2.25 | 2 | 5.76 | 5.68 | 5.720 | 1.302 | | |
| | 20 | 2.04 | 2.03 | 2.035 | 15.270 | —0.722 | 66.243 |
| 2.75 | 3 | 5.89 | 5.86 | 5.875 | 2.345 | | |
| | 24 | 2.24 | 2.24 | 2.240 | 20.079 | —0.671 | 104.117 |
| 3.25 | 3 | 6.97 | 6.89 | 6.930 | 2.521 | | |
| | 28 | 2.43 | 2.44 | 2.435 | 24.670 | —0.599 | 149.827 |

In order from this to find the pressure on the unit of surface, or $k$, we have to assume the lever arm $a=0.81705$ inch, as already shown above. The following table contains the values of $R$, as well as the reduced $r$, and the surfaces of the disks $F$, as to which latter it is to be noticed that after more careful measurements the radii of the second and third disks resulted 1.745 and 2.245:

TABLE I.

| $\rho$ | 1.25 | 1.745 | 2.245 | 2.75 | 3.25 |
|---|---|---|---|---|---|
| $R$ | 97.252 | 97.754 | 98.256 | 96.760 | 99.260 |
| * Reduced $r$ | 18.791 | 38,463 | 66 165 | 104.095 | 149.942 |
| $F$ | 4.909 | 9.566 | 15.834 | 23.758 | 33.182 |
| $k$ | 2.2700 | 2.3476 | 2.4028 | 2 4810 | 2.5199 |

* i. e., Reduced to standard density of air.

In order to avoid too small numbers these values of $k$ are given too large, and must be divided by one million in order to present the desired constant factors, which, multiplied by the squares of the velocities in inches, will give the pressures in loths for each square inch of the disk. This same multiplication of $k$ is also continued in the following paragraphs.

Many days later I repeated these observations with the same disks. The results were—

| $\rho$ | $G^1$ | $t_1$ | $t_2$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|
| 1.25 | 1 | 5.00 | 5.02 | 5.01 | 0.168 | —0.592 | 19.091 |
| | 10 | 1.91 | 1.91 | 1 91 | 4.641 | | |
| 1.75 | 1.5 | 5.21 | 5.22 | 5.215 | 0.721 | | |
| | 16 | 1.87 | 1.87 | 1.87 | 10.405 | —0.708 | 38.861 |
| 2.25 | 2 | 5.67 | 5.70 | 5.685 | 1.329 | | |
| | 20 | 2.05 | 2.04 | 2.045 | 15.291 | —0.746 | 67.066 |
| 2.75 | 3 | 5.74 | 5.79 | 5.765 | 2.338 | | |
| | 24 | 2.23 | 2.24 | 2.235 | 20.025 | —0.790 | 103.983 |
| 3.25 | 4 | 6.06 | 6.09 | 6.075 | 3.391 | | |
| | 28 | 2.45 | 2.43 | 2.44 | 24.633 | —0.694 | 150.786 |

The values of $R$ and $F$ are the same as in the first series. The following values of $k$ are computed from the reduced $r$:

TABLE II.

| $p$ | 1.25 | 1.745 | 2.245 | 2.75 | 3.25 |
|---|---|---|---|---|---|
| Reduced $r$ | 18.952 | 38.576 | 66.575 | 103.221 | 149.683 |
| $k$ | 2.2894 | 2.3549 | 2.4176 | 2.4602 | 2.5154 |

It evidently results that $k$ becomes larger as soon as the surface of the disk increases, as also that the differences are proportional, not to the increase of the surfaces, but to the increase of the radii.

Measures were also made with square disks whose sides measured $b = 2, 3, 4, 5, 6$ inches, respectively. These gave—

| $b$ | $G^1$ | $t_1$ | $t_2$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.5 | 5.80 | 5.86 | 5.830 | −0.188 | | |
| | 10 | 1.84 | 1.83 | 1.835 | +4.104 | −0.660 | 16.042 |
| 3 | 1 | 6.00 | 5.95 | 5.975 | +0.346 | | |
| | 14 | 1.97 | 1.96 | 1.965 | 8.840 | −0.684 | 36.774 |
| 4 | 2 | 6.06 | 6.03 | 6.045 | 1.364 | | |
| | 20 | 2.08 | 2.08 | 2.080 | 15.383 | −0.519 | 68.798 |
| 5 | 3 | 5.99 | 6.06 | 6.025 | 2.364 | | |
| | 24 | 2.30 | 2.28 | 2.290 | 20.168 | −0.643 | 109.135 |
| 6 | 4 | 6.50 | 6.43 | 6.465 | 3.443 | | |
| | 28 | 2.55 | 2.54 | 2.545 | 24.874 | −0.488 | 164.270 |

The closer investigation showed again that the surfaces of the disks in part needed some small corrections, as in the following Table III:

TABLE III.

| $b$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $R$ | 97.002 | 97.504 | 98.008 | 98.512 | 99.015 |
| Reduced $r$ | 15.607 | 35.810 | 67.053 | 106.455 | 160.522 |
| $F$ | 4.000 | 8.977 | 16.000 | 24.958 | 36.000 |
| $k$ | 2.3317 | 2.3472 | 2.4281 | 2.4338 | 2.5055 |

The following results were given by a subsequent repetition of the same observations:

| $b$ | $G^1$ | $t_1$ | $t_2$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.5 | 5.76 | 5.79 | 5.775 | −0.149 | −0.630 | 16.020 |
| | 10 | 1.84 | 1.83 | 1.835 | +4.128 | | |
| 3 | 1 | 5.96 | 5.94 | 5.950 | +0.397 | −0.641 | 36.744 |
| | 14 | 1.96 | 1.97 | 1.965 | 8.876 | | |
| 4 | 2 | 5.74 | 5.78 | 5.760 | 1.371 | −0.608 | 68,976 |
| | 20 | 2.07 | 2.07 | 2.070 | 15.387 | | |
| 5 | 3 | 5.92 | 5.93 | 5.925 | 2.415 | −0.714 | 109.855 |
| | 24 | 2.29 | 2.29 | 2.290 | 20.233 | | |
| 6 | 4 | 6.26 | 6.26 | 6.260 | 3.485 | −0.700 | 164.000 |
| | 28 | 2.53 | 2.53 | 2.530 | 24.922 | | |

According to this, the values of $k$ are:

TABLE IV.

| $b$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Reduced $r$ .. | 15.704 | 35.998 | 67.524 | 107.493 | 160.378 |
| $k$ .. | 2.3461 | 2.3595 | 2.4452 | 2.4574 | 2.5032 |

By connecting among themselves the two first, as also the two last series of observations, the law according to which the value of $k$ depends on the size of the disk may be approximately recognized, but the relation between the two forms of disks does not appear clearly. In order to discover this I tried allowing circular and square disks to run one immediately after the other, the radius of the first being 0.5 greater than the side of the latter. From this, however, it could only be inferred that for equal areas the resistance of the square disk is the greater.

In order to recognize the influence of the shape, I tried also disks which formed equilateral triangles of 7.6 inches on each side, which were fastened in such a way that one of the sides stood vertically at the end of an arm. The area of each disk measured 25 square inches, agreeing, therefore, to within a very small quantity, which subsequent accurate measures showed, with that of the square disk of 5 inches on a side. As I observed these two pair of disks one immediately after the other under the same load, it appeared that the square disk revolved somewhat more rapidly. This result, however, was not decisive, in that the distances of the centers of pressure from the axis of rotation, or $R$, did not remain the same. In this respect it may be mentioned that when the side of the equilateral triangle $=b$ and its altitude $=h=b \cos 30°$ and the distance of the center of the surface from the axis of rotation is $A$, we then find

$$R=\sqrt{(A^2+\tfrac{1}{18}h^2)}$$

A complete series of observations, together with the preliminary and the concluding determinations of the value of $p$ and $s$, gave the following:

| $G^1$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|
| 3 | 5.91 | 2.220 | .......... | ............ |
| 6 | 4.35 | 4.715 | .......... | ............ |
| 10 | 3.43 | 8.081 | .......... | ............ |
| 28 | 2.12 | 23.525 | −0.875 | +108.640 |

After the computation of $R=98.204$, as also after the reduction of $F$ and $r$ there is found

$$K=2.5026.$$

Directly following the above, the same observations were repeated with the square disk of 5 inches on a side with the following results:

| $G^1$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|
| 3 | 5.96 | 2.234 | ........ | ........ |
| 6 | 4.40 | 4.739 | ........ | ........ |
| 10 | 3.46 | 8.110 | ........ | ........ |
| 28 | 2.10 | 23.448 | −0.875 | +107.390 |

From these latter there finally resulted

$$k = +2.4491.$$

The results thus far obtained warrant the suspicion that for equal areas of the disks, the resistance becomes smaller the shorter is the deviated path that the air must describe in order to pass around the disk. Hence it is to be expected that the resistance would become especially small for long and narrow disks. Consequently I took a pair of disks 1 inch broad and 16 inches high, which therefore had the same area as the square disks of 4 inches on a side. These I allowed to run interchangeably with the square disks and under equal loads, but most unexpectedly the velocity of the square disks was always somewhat greater than the narrow ones. This was so much the more remarkable as the square ones, on account of the greater distances from the axis of rotation, were expected to show a greater resistance.

As at first I allowed these long disks to run under only two different loadings, I found

| $G^1$ | $t$ | $t$ | $t$ | $G$ | $z$ | $r$ |
|---|---|---|---|---|---|---|
| 2 | 6.33 | 6.69 | 6.51 | 1.514 | ........ | ........ |
| 20 | 2.07 | 2.09 | 2.08 | 15.488 | −0.075 | 67.332 |

For the feeble load the velocity had shown very discrepant values. Therefore the repetition of the observation was important, and for greater security this was done on the following day for six different loads.

| $G^1$ | $t$ | $G$ | $A$ | Diff. |
|---|---|---|---|---|
| 1 | 8.51 | 0.748 | 0.743 | −0.005 |
| 2 | 6.28 | 1.538 | 1.508 | −0.030 |
| 4 | 4.48 | 3.049 | 3.127 | +0.078 |
| 8 | 3.23 | 6.254 | 6.174 | −0.080 |
| 16 | 2.28 | 12.495 | 12.564 | +0.069 |
| 24 | 1.87 | 18.790 | 18.760 | −0.030 |

From this there results as the most probable values

$$z = -\ 0.171$$
$$r = +66.199$$

If these constants are introduced into the expression for $G$, the latter assumes the values given the column headed $A$, whose departures from the observed $G$ are given in the last column.

The surfaces of these disks measured very accurately 16 square inches, and the distance of the center of pressure was 96.500 inches. After reduction to the adopted normal density of the air the constants $r$ for the two series of observations became respectively

66.65 and 66.373

whence

$$k = 2.5286 \text{ and } k = 2.5178 \text{ respectively.}$$

The constant coefficient of the square of the velocity resulted therefore in this case as great as the series of observations III and IV would have led us to conclude would have been found for square disks of about 7 inches on each side; consequently the suspicion arises that the increase in the value of $k$ is not proportional to any linear dimension, but to the circumference of the disk. A simple consideration leads to the same result.

All previously given observations show that a disk of an area $F$ moving with a velocity $c$ through the air in. a direction normal to its plane experiences a resistance

$$D = k\ Fc^2.$$

If we analyze $k$ into two terms

$$k = \alpha + p\ \beta$$

where $p$ expresses the circumference of the disk, then the first part of $D$, namely, $\alpha\ Fc^2$, corresponds to the ordinary assumption. The second part

$$p\ Fc^2\ \beta = Fc.\ p.\ c.\ \beta$$

contains, as a factor, the mass of the passing air, which is proportional to $Fc$, also $p$, or the circumference of the disk, which the air touches. and finally the velocity $c$, under which this contact takes place. It appears therefore that the cause of the increase of the resistance can be none other than the friction of the air against the edge of the disk. However, as the experiments already mentioned in the preface have shown, the air immediately adjacent to the edge of the disk flows perfectly regularly past it, without taking up any whirling motion, which latter first forms behind where the air protected by the obstacle is touched. Friction is therefore (in accordance with the experience* with water) proportional to the first power of the velocity.

Before I computed the appropriate constants by the combination of all of the observations, I made an attempt to compare among them-

* "On the Influence of the Temperature on the Movement of Water in Tubes." Hagen, *Math. Abh. Akad. Wiss. Berlin*, 1854, p. 69.

selves the twenty-one observations made with circular and square disks, in order to convince myself as to what assumed value of $p$ presented the greatest agreement.

If I assumed for $p$ the circumference of the disk, there resulted

$$\alpha = 2.210$$
$$\beta = 0.0132$$

and the sum of the squares of the outstanding errors was

$$[xx] = 0.01425$$

By introducing the square root of the surface I obtained

$$\alpha = 2.200$$
$$\beta = 0.0526$$
$$[xx] = 0.00976$$

I then put $p$, equal to three different transverse lines drawn through the center of the disk. First, the smallest transversals, for which of course the sides of the square and the diameters of the circles were directly introduced. This gave

$$\alpha = 2.204$$
$$\beta = 0.0487$$
$$[xx] = 0.01282$$

For the greatest transversals, namely, the diagonals of the squares and diameters of the circles, I obtained

$$\alpha = 2.230$$
$$\beta = 0.0354$$
$$[xx] = 0.02221$$

Finally, for the average transversals which I drew [centrally] across the disks at distances apart of every 3 degrees, and took the arithmetical mean of all, I found

$$\alpha = 2.200$$
$$\beta = 0.04675$$
$$[xx] = 0.00966$$

It is evident that this latter method must lead to very nearly the same result as the introduction of the square root of the surface since $\beta$ diminishes in the same ratio as the coefficient of $\beta$ increases.

Judging by the sums of the squares of the errors it would, according to this, be advisable to introduce the square roots of the surfaces as factors, but this is impossible, even although the results of the observations made with the long disk should be included under this same law. There only remains to introduce the circumference as a factor, even although in this case notable departures still remain. These are in nowise however errors of observation, but result principally from the inevitable variations in friction. An error of 1 per cent. in the time could scarcely have been made, but still such discrepancies and even larger ones show themselves very frequently since the friction induced now faster and now slower motion. Nevertheless, from the following collection of all the observations it results that these have led to a quite trustworthy result.

| Radii and sides. | | $k$ | $P$ | $A$ | Diff. | Squares. |
|---|---|---|---|---|---|---|
| Circle $\rho=$ | 1.25 | 2.270 | 7.854 | 2.338 | +0.068 | 0.004624 |
| | 1.75 | 2.348 | 10.996 | 2.368 | +0.020 | 0400 |
| | 2.25 | 2.403 | 14.137 | 2.397 | −0.006 | 0036 |
| | 2.75 | 2.481 | 17.279 | 2.427 | −0.054 | 2916 |
| | 3.25 | 2.520 | 20.420 | 2.456 | −0.064 | 4096 |
| Circle $\rho=$ | 1.25 | 2.289 | 7.854 | 2.338 | +0.049 | 2401 |
| | 1.75 | 2.355 | 10.996 | 2.368 | +0.013 | 0169 |
| | 2.25 | 2.418 | 14.137 | 2.397 | −0.021 | 0441 |
| | 2.75 | 2.460 | 17.279 | 2.427 | −0.033 | 1089 |
| | 3.25 | 2.515 | 20.420 | 2.456 | −0.059 | 3481 |
| Square $b=$ | 2. | 2.332 | 8.0 | 2.339 | +0.007 | 0049 |
| | 3. | 2.347 | 12.0 | 2.377 | +0.030 | 0900 |
| | 5. | 2.428 | 16.0 | 2.415 | −0.013 | 0169 |
| | 5. | 2.434 | 20.0 | 2.452 | +0.018 | 0324 |
| | 6. | 2.505 | 24.0 | 2.490 | −0.015 | 0225 |
| Square $b=$ | 2. | 2.346 | 8.0 | 2.339 | −0.007 | 0049 |
| | 3. | 2.360 | 12.0 | 2.377 | +0.017 | 0289 |
| | 4. | 2.445 | 16.0 | 2.415 | −0.030 | 0900 |
| | 5. | 2.457 | 20.0 | 2.452 | −0.007 | 0049 |
| | 6. | 2 503 | 24.0 | 2.490 | −0.013 | 0169 |
| Triangle.......... | | 2.503 | 22.795 | 2.479 | −0.024 | 0576 |
| Square $b=$ | 5. | 2.449 | 20.0 | 2.452 | +0.003 | 0009 |
| Parallelogram .... | | 2.529 | 34.0 | 2.584 | +0.055 | 3025 |
| Parallelogram .... | | 2.518 | 34.0 | 2.584 | +0.066 | 4356 |
| | | | | | | 0.030742 |

From this table there results as the most probable values

$$\alpha=2.2639$$
$$\beta=0.009416$$

The values of $k$ computed from this are given in the column $A$; from the differences in the next column, with reference to the observed values of $k$, there results the probable error 0.0252, and we find the probable error of $\alpha$ equal to 0.01338, or about $\frac{1}{2}$ per cent., and of $\beta$ equal to 0.000719, or about $7\frac{1}{2}$ per cent.

Although the reliability of these results, especially in their application to still larger surfaces and greater velocities, leaves much to be desired, still scarcely any important higher degree of accuracy is to be attained with apparatus that is similar to that above described. On the other hand the concluded law of resistance would be in an important degree confirmed or corrected, if on a firm rod in front of a locomotive, disks are fastened, whose pressure could be measured by the tension of a spring, while the milestones on the roadside would serve very conveniently for the determination of the velocity.*

* [This experiment has been carried on recently by Wild and others, but the resulting value of $k$ is not so reliable as that deduced from observations with large whirling machines.—*C. A.*]

From the preceding it results that the pressure of the air against a plane disk turned normally towards it is

$$D = \frac{2.264 + 0.00942 \times p}{1{,}000{,}000} F c^2$$

Where $D$ is expressed in old Prussian loths and $p$, $F$, and $c$ in [Rhenish] inches. According to the above, the pressure against a square disk of 1 square foot area, moving with a velocity of 50 feet per second, would for example be 140.8 loths, or nearly 4.4 pfund.

For reduction to metric measures and weights I take not the metre itself but the decimetre as the unit of measure for lengths and surfaces, in order to remain within the limits of the observations. Therefore the resistance of the air for a temperature of 15° C. and a barometric pressure of 28 Paris inches,* expressed in grammes, amounts to

$$(0.00707 + 0.0001125\,p)\,F c^2,$$

Where $p$ represents the circumference of the disk, $F$ the sectional area, and $c$ the velocity expressed in decimetres.

The pressure that very small disks experience when struck normally by a current of air is also given by another simple consideration, whose correctness has in general been confirmed by many experiments. These experiments indeed are limited, so far as known, to streams of water; but the expansibility of the air is certainly in this case without influence, since the observations mentioned in the preface, upon the direction and strength of currents deviated in front of opposing disks, showed identical results with water and with air.

Imagine a vessel filled to the height $h$ with a fluid of which one unit of volume or 1 cubic inch weighs $\gamma$ loths. The bottom of the vessel therefore experiences on each square inch a pressure equal to $\gamma h$, when no side pressure exists. If there is suddenly made therein an opening of 1 square inch, the outflow of the fluid through it begins with the velocity $c = 2\sqrt{gh}$†, and if we catch the stream by an equally large surface directed normally against it, then the pressure $D$ upon this is again equally as great as before upon the bottom of the vessel, namely, $\gamma h$. From this we have

$$D = \gamma h = \frac{\gamma}{4g} c$$

For the density of the air above adopted its specific weight is 0.001223; therefore a cubic inch weighs 0.001495 loth, and $g$ is equal to 187.6, if the semi-acceleration due to gravity is expressed in inches. From this we have these results:

$$D = 0.000001992 = 1.992 \text{ millionths of a loth.}$$

* The density is that of air at 15° C. and 28 Paris inches or 757.96$^{mm}$ under gravity at Berlin (52° 30′), but strictly speaking the pressure should be stated in standard measure as 758.47$^{mm}$ under gravity at 45° and sea level.

† $g$ is the height fallen through in 1 second, or one-half the acceleration due to gravity.

As the first term of the above value of $k$ comes out 2.264 or larger than this by nearly 14 per cent., the stronger resistance deduced from the observations is explained by the rarefaction of the air occurring at the rear of the disk, which rarefaction in the case of an assumed outflow into empty space does not take place.

Although the present investigation is confined only to those positions in which the disks are turned normal to the direction of their motion, still it was important to be convinced that slight and unavoidable deviations from this normal position had no important influence. The pins by means of which the disks were fastened to the arms were directed radially towards the axis of rotation. Thus the disks could be given any desired inclination to the direction of their motion. One such experience however showed this arrangement to be entirely unallowable in the observations, in that the simple relation between the resistance and the velocity of the disk completely disappeared. The reason for this irregularity is apparent. According as the two disks were inclined downwards or upwards they were pressed up or down by the impinging air, and by so much the more the greater their velocity was. The arms with the inclined disks and with the axis of rotation therefore pressed variably upon the steel point on which the axis rested, and accordingly the screw threads on the axis were variably pressed up or down, whereby the friction each time experienced an important change. When however I inclined one disk upwards and the opposite disk downwards, the axis was pressed to one side, and by so much the more, the greater the velocity was.

In order not to change the simple arrangements for fastening the disks, I provided the two 5-inch square disks with roof shaped piece, in addition, so that in front of the lower half of the disk the inclined plane was turned upwards, and in front of the lower half an equal plane with the same inclination was turned downwards. Each of the two disks thus changed was thus both raised and depressed by equal forces for all velocities, so that the injurious effect upon the axis of rotation disappeared.

A complete series of observations (wherein both at the beginning and at the end the arms were set in motion without disks in order to determine the resistance) gave—

(*a*) When the roof surface was inclined 40° to the vertical or to the plane disk,

$$r = 83.92.$$

(*b*) For an inclination of 20° to the vertical,

$$r = 101.16.$$

(*c*) And for the plane disk itself, therefore, after removing the additions

$$r = 110.93.$$

If we divide these values by the cosines of 40, 20, and 0 degrees, respectively, there results

109.55, 107.65, and 110.93.

The resistances are therefore in accordance with the ordinary assumption, proportional to the cosine of the inclination.

In case the plane of the plane disk does not include the axis of rotation, we should also have to consider the diminution of the surface opposed to the impinging air in consequence of the projection upon the direction of motion, and for both reasons the resistance diminishes in the ratio of the square of the cosine of the deviation. Since the disks were always adjusted by the plumb line, therefore an error of 2 degrees, by which the resistance would only be diminished by its thousandth part, could not easily remain unnoticed.

Finally, it still remains to be investigated whether the nature of the surface of the disks, according as they were smooth or rough, had any influence on the resistance. To this end I took two disks, each of which was covered on one side with very smooth paper but on the other with very coarse sandpaper. I allowed these to run with various velocities, exposing each time first the smooth and then the rough side to the impinging air. In both cases the times in which the index described 10 inches remained very nearly the same. The differences were very irregular, and not larger than occurred in repeated experiments with equal pairs of disks. Hence the nature of the surface of a plane disk has no influence on the resistance of the air when the surfaces are normal to the direction of motion.

II.

# ON THE INTEGRALS OF THE HYDRO-DYNAMIC EQUATIONS THAT REPRESENT VORTEX-MOTIONS.*

By Prof. HERMANN VON HELMHOLTZ.

Hitherto the integrals of the hydro-dynamic equations have been sought almost exclusively under the assumption that the rectangular components of the velocity of every particle of liquid can be put equal to the differential quotients in the corresponding directions of a certain definite function that we will call the velocity potential.

On the one hand Lagrange† had proven that this assumption is allowable whenever the movement of the mass of water has arisen and is maintained under the influence of forces that can be expressed as the differential quotients of a force potential, and even that the influence of moving solid bodies that come in contact with the liquid do not affect the applicability of the assumption. Since now most of the forces of nature that are easily expressed mathematically can be presented as the differential quotients of a force potential, therefore also by far the majority of the cases of fluid motion that are treated mathematically fall into the category of those for which a velocity potential exists.

On the other hand, even Euler ‡ had called attention to the fact that there are cases of fluid motion where no velocity potential exists; *e. g.*, the rotation of a fluid with equal angular velocities in all its parts about an axis. The magnetic forces that act upon a fluid permeated by electric currents, and especially the friction of fluid particles on each other and on solid bodies, belong to the forces that can give rise to such forms of motion. The influence of friction on fluids could not hitherto be mathematically defined, and yet it is very large in all cases where we are not treating of infinitely small vibrations, and causes the most important deviations between theory and nature. The difficulty of defining this influence and of finding methods for its measurement certainly lay

* Crelle's *Journal für die reine und angewandte Mathematik*, 1858, vol. LV, p. 25–85. Helmholtz, *Wissenschaftliche Abhandlungen*, 1882, vol. I, pp. 101–134. *London, Edinburgh, and Dublin Philosophical Magazine*, June, 1867 (4), XXIII. pp. 485–510

† *Mécanique Analytique*, Paris, 1815, vol. II, p. 304.

‡ *Histoire de l'Académie des Sciences de Berlin*, anno 1755, p. 292.

mostly in the fact that we had no idea of the forms of motion that friction produces in the fluid. Therefore in this respect an investigation of those forms of motion in which no velocity potential exists seems to me to be of importance.

The following investigation will now show that in those cases in which a velocity potential does exist the smallest particles of liquid have no motion of rotation, but that when no velocity potential exists then a part at least of the liquid particles are in the act of rotation.

By *vortex lines* (Wirbellinien) I designate lines that are so drawn through the mass of liquid that their directions everywhere coincide with the direction of the instantaneous axis of rotation of the liquid particles at that point of the line.

By *vortex filaments* (Wirbelfäden) I designate the portion of the mass of liquid that is cut out when we construct the corresponding vortex lines passing through every point of the circumference of an infinitely small element of the surface.

The following investigation shows that when a force potential exists for all the forces that act upon the fluid then:

(1) No particle of liquid acquires rotation that was not in rotation from the beginning.

(2) The particles of liquid that at any moment belong to the same vortex line remain belonging to the same vortex line, even although they have a motion of translation.

(3) The product of the sectional area by the velocity of rotation of an infinitely slender vortex filament is constant along the whole length of the filament and also retains the same value during the translatory motion of the filament. Therefore the vortex filaments must return into themselves within the liquid or can only have their ends at the boundaries of the fluid.

This last proposition makes it possible to determine the velocities of rotation when the form of a particular vortex filament is given at different moments of time. Further we solve the problem to determine the velocity of the particles of liquid for a given moment of time when the velocities of rotation are given for this moment, but in the solution there remains undetermined one arbitrary function that must be utilized to satisfy the boundary conditions.

This last problem leads to a remarkable analogy between the vortex motions of liquids and the electro-magnetic actions of electric currents.

When in a simply connected space* filled with moving liquid a velocity potential exists, the velocities of the liquid particles are equal to and in the same direction as the forces that a certain distribution of

* I use this expression (einfach zusammenhängenden Raume) in the same sense in which Riemann (*Journal für die reine und angewandte Mathematik*, 1857, LIV, p. 108) speaks of simple and multiple-connected surfaces. A space that is $n$-times connected is therefore one such that $n$—1 but not more intersecting surfaces can pass through it without cutting the space into two completely separate portions. A ring is therefore in this sense a doubly-connected space. The intersecting surfaces must be completely surrounded by the lines in which they cut the surface of the space.

magnetic masses on the surface of the space would exert upon a magnetic particle in the interior.

On the other hand, when vortex threads exist in any such space the velocities of the liquid particles are equal to the forces exerted upon a magnetic particle by a closed electric current that flows partly through the vortex filaments in the interior of the mass and partly in the boundary surface, and whose intensity is proportional to the product of the sectional area of the vortex filament by its velocity of rotation.

I shall therefore in the following lines often allow myself to hypothecate the presence of magnetic masses or of electric currents, simply in order thereby to obtain shorter and more perspicuous expressions for the nature of functions that are just the same functions of the coördinates as the potential functions, or the attractive forces for a magnetic particle, are of the magnetic masses or electric currents.

By these propositions the forms of motion concealed in that class of integrals of the hydro-dynamic equations not hitherto treated of become accessible at least to the imagination even although it be possible to execute the complete integration only in a few of the simplest cases where only one or two rectilinear or circular vortex filaments are present in masses of liquid that are either unlimited or partially bounded by one infinite plane.

It can be demonstrated that rectilinear parallel vortex filaments in a mass of water that is bounded only by planes perpendicular to such filaments, rotate about their common center of gravity, when in the determination of this center we consider the velocity of rotation as equivalent to the density of a mass. In this rotation the location of the center of gravity remains unchanged. On the other hand, for circular vortex filaments, all standing perpendicular to a common axis, the center of gravity of their cross-section advances parallel to the axis.

## I. DEFINITION OF ROTATION.

At a point within a liquid whose position is defined by the rectangular coördinates $x$, $y$, $z$, and at the time $t$, let the pressure be $p$, the three components of the velocity $u$, $v$, $w$, the three components of the external forces acting on the unit mass of the liquid $X$, $Y$, $Z$, and $h$ be the density whose changes can be considered as negligible; the established equations of motion for an interior point of the fluid are:

$$\left.\begin{aligned}
X-\frac{1}{h}\frac{\partial p}{\partial x}&=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\\
Y-\frac{1}{h}\frac{\partial p}{\partial y}&=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\\
Z-\frac{1}{h}\frac{\partial p}{\partial z}&=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\\
0&=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Hitherto, almost exclusively, only those cases have been treated where not only the forces $X$, $Y$, $Z$, have a potential $V$ so that they can be expressed in the form,

$$X=\frac{\partial V}{\partial x}, Y=\frac{\partial V}{\partial y}, Z=\frac{\partial V}{\partial z}, \quad \ldots \ldots \quad (1a)$$

but also where a velocity potential $\varphi$ can be found so that

$$u=\frac{\partial \varphi}{\partial x}, \ v=\frac{\partial \varphi}{\partial y}, \ w=\frac{\partial \varphi}{\partial z}, \quad \ldots \ldots \quad (1b)$$

The problem is thereby greatly simplified since the first three of equations (1) give a common integral equation from which to find $p$ after we have determined $\varphi$ in accordance with the fourth equation which in this case takes the form

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 \varphi}{\partial z^2}=0,$$

and which therefore agrees with the established differential equation for the potential of magnetic masses that lie outside the space within which this equation holds good. Moreover, it is known that every function $\varphi$ that satisfies this last differential equation within a simply connected space,* can be expressed as the potential of a definite distribution of magnetic masses on the boundary surface of the space as I have stated already in the introduction.

In order that we may be able to make the substitution required in the equation (1b) we must have

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0, \ \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=0, \ \frac{\partial w}{\partial x}-\frac{\partial v}{\partial z}=0, \quad \ldots \ldots \quad (1c.)$$

In order to understand the mechanical significance of these last three conditions, we may imagine the change that any infinitely small volume of water experiences in the elementary time $dt$ to be compounded of three different motions: (1) a motion of transference of the whole through space: (2) an expansion or contraction of the particle along the axis of dilatation, whereby every rectangular parallelopipedon of water whose sides are parallel to the principal axis of dilatation remains rectangular while its sides change their lengths but remain parallel to their original directions: (3) a rotation about some temporary axis of rotation having any given direction, which rotation can by a well-known proposition be always considered as the resultant of three rotations about the three coördinate axes.

*In manifold-connected-spaces $\phi$ can have several values, but for such many valued functions as satisfy the above differential equations the fundamental proposition of Green's theory of electricity no longer holds good (see Crelle *Journal*, XLIV, p. 360, or "The Mathematical Papers of the late George Green"), and therefore fail also a greater part of the propositions resulting from this which Gauss and Green have demonstrated for the magnetic potential functions, which functions are in their very nature always uni-valued.

If the conditions (1$c$) are satisfied at the point whose coördinates are $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, and if we designate the values of $u$, $v$, $w$, and their differential quotients as follows:

$$u=A, \quad \frac{\partial u}{\partial x}=a, \quad \frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}=\alpha.$$

$$v=B, \quad \frac{\partial v}{\partial y}=b, \quad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}=\beta.$$

$$w=C, \quad \frac{\partial w}{\partial z}=c, \quad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}=\gamma.$$

We obtain for the point whose coördinates $x$, $y$, $z$, differ differentially from $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$:

$$u=A+a\,(x-\mathfrak{x})+\gamma\,(y-\mathfrak{y})+\beta\,(z-\mathfrak{z}),$$

$$v=B+\gamma\,(x-\mathfrak{x})+b\,(y-\mathfrak{y})+\alpha\,(z-\mathfrak{z}),$$

$$w=C+\beta\,(x-\mathfrak{x})+\alpha\,(y-\mathfrak{y})+c\,(z-\mathfrak{z}),$$

or when we put:

$$\begin{aligned}\varphi=&A\,(x-\mathfrak{x})+B\,(y-\mathfrak{y})+C\,(z-\mathfrak{z})\\&+\tfrac{1}{2}\,a\,(x-\mathfrak{x})^2+\tfrac{1}{2}b\,(y-\mathfrak{y})^2+\tfrac{1}{2}c\,(z-\mathfrak{z})^2\\&+\alpha(y-\mathfrak{y})\,(z-\mathfrak{z})+\beta\,(x-\mathfrak{x})\,(z-\mathfrak{z})+\gamma\,(x-\mathfrak{x})\,(y-\mathfrak{y}),\end{aligned}$$

there results:

$$u=\frac{\partial\varphi}{\partial x},\quad v=\frac{\partial\varphi}{\partial y},\quad w=\frac{\partial\varphi}{\partial z}$$

It is well known that by a proper selection of another system of rectangular coördinates $x_1$, $y_1$, $z_1$, whose origin is at the point $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, the expression for $\varphi$ can be brought into the form:

$$\varphi=A_1\,x_1+B_1\,y_1+C_1\,z_1+\tfrac{1}{2}\,a_1\,x_1^2+\tfrac{1}{2}b_1\,y_1^2+\tfrac{1}{2}\,c_1\,z_1^2$$

where the component velocities $u_1$, $v_1$, $w_1$, along these new coördinate axes have the values:

$$u_1=A_1+a_1\,x_1,\qquad v_1=B_1+b_1\,y_1,\qquad w_1=C_1+c_1\,z_1.$$

The velocity $u_1$ parallel to the axis of $x_1$ is therefore alike for all liquid particles that have the same value of $x_1$, therefore particles that at the beginning of the elementary time $dt$ lie in a plane parallel to that of $y_1\,z_1$ are also still in that plane at the end of the elementary time $dt$. This same proposition is true for the planes $x_1\,y_1$ and $x_1\,z_1$. Therefore when we imagine a parallelopipedon bounded by three planes parallel to the last named coördinate planes and infinitely near to them, the liquid particles inclosed therein still form at the end of the time $dt$ a rectangular parallelopipedon whose surfaces are parallel to the same coördinate planes. Therefore the whole motion of such an indefinitely small parallelopipedon is, under the assumption expressed

in (1$c$) compounded only of a motion of translation in space and an expansion and contraction of its edges and it has no rotation.

We return now to the first system of coördinates, that of $x, y, z$, and imagine added to the hitherto existing motion of the infinitely small mass of liquid surrounding the point $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, a system of rotatory motions about axes that are parallel to those of $x, y, z$, and that pass through the point $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, and whose angular velocities of rotation may be $\xi, \eta, \zeta$, thus then the component velocities parallel to the coördinate axes of $x, y, z$, as resulting from such rotations are respectively:

| Parallel to $x$: | Parallel to $y$: | Parallel to $z$: |
|---|---|---|
| o, | $(z-z)\ \xi$, | $-(y-\mathfrak{y})\ \xi$, |
| $-(z-\mathfrak{z})\ \eta$, | o, | $(x-\mathfrak{x})\ \eta$, |
| $(y-\mathfrak{y})\ \zeta$, | $-(x-\mathfrak{x})\ \zeta$, | o. |

Therefore the velocities of the particles whose coördinates are $x, y, z$, become:

$$u=A+a(x-\mathfrak{x})+(\gamma+\zeta)\ (y-\mathfrak{y})+(\beta-\eta)\ (z-\mathfrak{z}),$$

$$v=B+\ (\gamma-\zeta)\ (x-\mathfrak{x})+b\ (\mathfrak{x}-y)+(\alpha+\xi)\ (z-\mathfrak{z}),$$

$$w=C+\ (\beta+\eta)\ (x-\mathfrak{x})\ +\ (\alpha-\xi)\ (y-\mathfrak{y})+c\ (z-\mathfrak{z}),$$

whence by differentiation there results:

$$\left.\begin{aligned} \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}&=2\xi. \\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}&=2\eta. \\ \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}&=2\zeta. \end{aligned}\right\} \quad \text{. . . . . . . . . .} \quad (2)$$

Therefore the quantities on the left-hand side, which according to equation (1$c$) must be equal to zero in order that a velocity potential may exist, are equal to double the velocity of rotation about the three coördinate axes of the liquid particles under consideration. The existence of a velocity potential excludes the existence of a rotary motion of the particles of liquid.

As a further characteristic peculiarity of fluid motions that have a velocity potential, it may be further stated that in a simply-connected space $S$, entirely inclosed within rigid walls and wholly filled with fluid, no such motion can occur; for when we indicate by $n$ the normal directed inwards to the surface of such space then the component velocity $\frac{\partial \varphi}{\partial n}$ directed perpendicular to the wall must be everywhere

equal to zero. Therefore, according to the well-known Green's theorem,*

$$\iiint \left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right] dx\,dy\,dz = -\int \varphi \frac{\partial\varphi}{\partial n} d\omega,$$

where, on the left hand, the integration is to be extended over the whole of the volume $S$, but on the right hand over the whole surface $S$ whose elementary surface is designated by $d\omega$. If, now, $\frac{\partial\varphi}{\partial n}$ is to be equal to zero for the whole surface, then the integral on the left hand must also be zero, which can only be true when for the whole volume $S$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial z}=0,$$

that is to say, when there exists no motion whatever of the liquid. Every motion within a simply connected space of a limited mass of fluid that has a velocity potential is therefore necessarily connected with a motion of the surface of the fluid. If this motion of the surface, *i. e.*, $\frac{\partial\varphi}{\partial n}$, is known completely, then the whole movement of the inclosed fluid mass is also thereby definitely determined. For suppose there are two functions, $\varphi_{,}$ and $\varphi_{,,}$, that simultaneously satisfy the equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

in the interior of the space $S$, and also the condition

$$\frac{\partial\varphi}{\partial n}=\psi$$

for the surface of $S$, where $\psi$ indicates the value of $\frac{\partial\varphi}{\partial n}$ deduced from the assumed motion of the surface, then would the function $(\varphi_{,}-\varphi_{,,})$ also satisfy the first condition for the interior of the space $S$, but for the surface this function would give

$$\frac{\partial(\varphi_{,}-\varphi_{,,})}{\partial n}=0;$$

whence, as just shown, it would follow that for the whole interior of $S$ we would have

$$\frac{\partial(\varphi_{,}-\varphi_{,,})}{\partial x}=\frac{\partial(\varphi_{,}-\varphi_{,,})}{\partial y}=\frac{\partial(\varphi_{,}-\varphi_{,,})}{\partial z}=0.$$

Therefore both functions would also correspond to exactly the same velocities throughout the whole interior of $S$.

Therefore rotations of liquid particles and circulatory motions within simply-connected wholly inclosed spaces can only occur when no velocity potential exists. We can therefore in general characterize the motions in which a velocity potential does not exist, as vortex motions.

*This is the proposition in Crelle *Journal*, vol. LIV, p. 108, already alluded to, and which does not hold good for complex or manifold-connected space.

## II. PERMANENCE OF THE VORTEX MOTION.

We will next determine the variations of the velocities of rotation $\xi$, $\eta$, $\zeta$ during the movement (of the surface) when only such forces are effective as have a force potential.

I note first in general that when $\psi$ is a function of $x$, $y$, $z$, $t$, and increases by the quantity $\delta\psi$, while the last four quantities increase by $\delta x$, $\delta y$, $\delta z$, and $\delta t$, respectively, we have:

$$\delta\psi = \frac{\partial\psi}{\partial t}\delta t + \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial z}\delta z.$$

If now the variation of $\psi$ during the short time $\delta t$ is to be determined for one and the same particle of liquid, we must give the quantities $\delta x$, $\delta y$, $\delta z$ the same values that they would have for the moving particle of liquid, namely:

$$\delta x = u\,\delta t, \qquad \delta y = v\,\delta t, \qquad \delta z = w\,\delta t,$$

and obtain:

$$\frac{\delta\psi}{\delta t} = \frac{d\psi}{dt} + u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} + w\frac{\partial\psi}{\partial z}$$

I shall in the following always use the notation $\frac{\delta\psi}{\delta t}$ only in the sense that $\frac{\delta\psi}{\delta t}dt$ indicates the variation of $\psi$ during the element of time $d\,t$ for the same particle of water whose coördinates at the beginning of the time $dt$ were $x$, $y$, and $z$.

If by differentiation we eliminate the quantity $p$ from the first of the equations (1) and introduce the notation of equations (2) and substitute for the forces $X$, $Y$, $Z$ the expressions in equation (1$a$), we obtain the following three equations:

$$\left.\begin{aligned}
\frac{\delta\xi}{\delta t} &= \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} \\
\frac{\delta\eta}{\delta t} &= \xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z} \\
\frac{\delta\zeta}{\delta t} &= \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z}
\end{aligned}\right\} \quad \cdots\cdots \quad (3)$$

or

$$\left.\begin{aligned}
\frac{\delta\xi}{\delta t} &= \xi\frac{\partial u}{\partial x} + \eta\frac{\partial v}{\partial x} + \zeta\frac{\partial w}{\partial x} \\
\frac{\delta\eta}{\delta t} &= \xi\frac{\partial u}{\partial y} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial w}{\partial y} \\
\frac{\delta\zeta}{\delta t} &= \xi\frac{\partial u}{\partial z} + \eta\frac{\partial v}{\partial z} + \zeta\frac{\partial w}{\partial z}
\end{aligned}\right\} \quad \cdots\cdots \quad (3a)$$

If $\xi$, $\eta$, and $\zeta$ for any particle of water are simultaneously zero then also—

$$\frac{\delta\xi}{\delta t} = \frac{\delta\eta}{\delta t} = \frac{\delta\zeta}{\delta t} = 0.$$

*Therefore those particles of water that do not already have a rotatory motion will receive none in the subsequent time.*

As is well known, we can combine rotations together after the method of parallelograms of forces. If $\xi$, $\eta$, $\zeta$ are the velocities of the rotations about the coördinate axes, then the velocity of rotation ($q$) about the instantaneous axis of rotation is

$$q=\sqrt{\xi^2+\eta^2+\zeta^2}$$

and the cosines of the angles that this axis makes with the coördinate axes are respectively $\frac{\xi}{q}$, $\frac{\eta}{q}$, and $\frac{\zeta}{q}$.

If now we consider an infinitely small distance $q\varepsilon$ in the direction of the instantaneous axis of rotation, then the projections of this distance on the three coördinate axes are respectively $\varepsilon\xi$, $\varepsilon\eta$, and $\varepsilon\zeta$. While at the point $x\ y\ z$ [at one end of $q\varepsilon$] the components of the velocity are $u$. $v$, $w$, they are at the other end of $q\varepsilon$ respectively

$$u_1=u+\varepsilon\xi\frac{\partial u}{\partial x}+\varepsilon\eta\frac{\partial u}{\partial y}+\varepsilon\zeta\frac{\partial u}{\partial z},$$

$$v_1=v+\varepsilon\xi\frac{\partial v}{\partial x}+\varepsilon\eta\frac{\partial v}{\partial y}+\varepsilon\zeta\frac{\partial v}{\partial z},$$

$$w_1=w+\varepsilon\xi\frac{\partial w}{\partial x}+\varepsilon\eta\frac{\partial w}{\partial y}+\varepsilon\zeta\frac{\partial w}{\partial z}.$$

Therefore in the course of the elementary time $dt$ the projection of the distance of the two particles of liquid that at the beginning of $dt$ were distant by the quantity $q\varepsilon$ has attained a value that, considering the equation (3), can be written as follows:

$$\varepsilon\xi+(u_1-u)\,dt=\varepsilon\left(\xi+\frac{\delta\xi}{\delta t}dt\right),$$

$$\varepsilon\eta+(v_1-v)\,dt=\varepsilon\left(\eta+\frac{\delta\eta}{\delta t}dt\right),$$

$$\varepsilon\zeta+(w_1-w)\,dt=\varepsilon\left(\zeta+\frac{\delta\zeta}{\delta t}dt\right).$$

On the left are the projections of the new location of the connecting line $q\varepsilon$; on the right are the projections multiplied by the constant factor $\varepsilon$ of the new velocity of rotation. It follows from these equations that the line connecting the two liquid particles that at the beginning of the time $dt$ limited the portion $q\varepsilon$ of the instantaneous axis of rotation will also after the lapse of the time $dt$ still coincide with the now changed axis of rotation.

When we, as above agreed on, call a line whose direction throughout its whole length agrees with the direction of the instantaneous axis of rotation of the particle of liquid at each point, a *vortex line*, we can express the proposition just found as follows: *Every vortex line remains permanently composed of the same particles of liquid while it progresses with these particles through the liquid.*

The rectangular components of the velocity of rotation increase in the same ratio as the projections of the portion $q\varepsilon$ of the axis of rotation; hence it follows that *the magnitude of the resulting velocity of rotation varies for a given particle of liquid in the same ratio as the distance of this particle from its neighbors in the axis of rotation.*

Imagine a vortex line drawn through all points of the circumference of an indefinite small surface. Then will a thread of infinitely small section, which is called the "vortex filament," be thereby cut out of the liquid. The volume of a portion of such a filament included between two given particles of liquid, which volume according to the propositions just proven always remains filled by the same particles, must remain constant during its progressive motion; therefore its section must vary in the inverse ratio of its length. Hence we can express the last proposition thus· *In a portion of a vortex filament, consisting of the same particles of liquid, the product of the velocity of rotation by the section ever remains constant during its translatory motion.*

From equation (2) it directly follows that—

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

Hence it further follows that—

$$\iiint\left(\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z}\right)dx\, dy\, dz=0$$

where the integration can be extended over any arbitrary portion $S$ of the mass of liquid. When we partially integrate this there results:

$$\int\int \xi\, dy\, dz+\int\int \eta\, dx\, dz+\int\int \zeta\, dx\, dy=0$$

where the integrations are to be extended over the whole surface of the volume of $S$. If we let $d\omega$ be an element of this surface and $\alpha, \beta, \gamma$ the three angles that the normal to $d\omega$ drawn outwards makes with the coördinate axes, then—

$$dy\, dz=\cos\alpha\, d\omega, \qquad dx\, dz=\cos\beta\omega, \qquad dx\, dy=\cos\gamma\, d\omega;$$

therefore

$$\int\int(\xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma)d\omega=0,$$

or when we let $\sigma$ be the resulting velocity of rotation and $\theta$ the angle between this velocity and the normal

$$\int\int\sigma\cos\theta\, d\omega=0,$$

where the integration is to be extended over the whole surface of $S$.

Let $S$ be a portion of a vortex filament bounded by two infinitely small planes $\omega_{,}$ and $\omega_{,,}$ perpendicular to the axis of the filament, then will $\cos\theta=+1$ for one of these planes and $\cos\theta=-1$ for the other, but $\cos\theta=0$ for the whole of the remaining surface of the filament; consequently, if $\sigma_{,}$ and $\sigma_{,,}$ are the velocities of rotation at $\omega_{,}$ and $\omega_{,,}$, respectively, the last equation reduces to

$$\sigma_{,}\,\omega_{,}=\sigma_{,,}\,\omega_{,,}$$

whence it follows that *the product of the velocity of rotation by the area of the section is constant throughout the whole length of the vortex filament.* It has already been shown that this product does not change during the progressive motion of the filament.

It follows from this that a vortex filament can not possibly end anywhere within the fluid, but must either return into itself, like a ring within the fluid, or must continue on to the boundaries of the fluid. For in case a filament ended anywhere within the fluid it would be possible to construct a closed surface for which the integral $\int \sigma \cos \theta \, d\omega$ is not zero.

### III. INTEGRATION BY VOLUME.

When we can determine the motions of the vortex filaments present in the fluid we can, by means of the above established propositions, also determine completely the quantities $\xi$, $\eta$, and $\zeta$. We will now consider the problem to determine the velocities $u$, $v$, and $w$ from the quantities $\xi$, $\eta$, and $\zeta$.

Within a mass of liquid that fills the region $S$ let values of $\xi$, $\eta$, and $\zeta$ be given, which quantities should satisfy the condition that—

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z}=0 \qquad (2a).$$

Such values of $u$, $v$, and $w$ are to be found as may, throughout the whole region $S$, satisfy the conditions [of Eq. ($1_4$) and (2) viz.]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1_4)$$

$$\left.\begin{aligned} \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} &= 2\xi \\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z} &= 2\eta \\ \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} &= 2\zeta \end{aligned}\right\} \qquad (2)$$

to which is still to be added the condition demanded by the boundary of the region $S$, according to the nature of the specific problem in hand.

According to the distribution of $\xi$, $\eta$, $\zeta$, as above specifically given, there can occur on the one hand such vortex lines as shall return into themselves within the limits of the region $S$ and on the other hand such as extend to the boundary and there suddenly break off. When this latter is the case then we can certainly prolong these [fragments of] vortex lines either along the surface of $S$ or beyond $S$ until they return into themselves, so that a larger space $S_1$ exists that contains only closed vortex lines and for whose whole surface both $\xi$, $\eta$, $\zeta$ and their resultant $\sigma$ itself are all equal to zero or at least

$$\xi \cos \alpha+\eta \cos \beta+\zeta \cos \gamma=\sigma \cos \theta=0.$$

Here, as before, $\alpha$, $\beta$, $\gamma$ indicate the angles between the coördinate axes and the normal to the appropriate portion of the surface of $S_1$.

$\theta$ indicates the angle between the normal and the resulting axis of rotation.

We now obtain the values of $u$, $v$, $w$, that satisfy the equations ($1_4$) and (2) by putting

$$\left.\begin{aligned} u &= \frac{\partial P}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \\ v &= \frac{\partial P}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \\ w &= \frac{\partial P}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

and determine the quantities $L$, $M$, $N$, $P$ by the conditions that within the region $S_1$ we must have

$$\left.\begin{aligned} \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} + \frac{\partial^2 L}{\partial z^2} &= 2\xi, \\ \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2} &= 2\eta, \\ \frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} + \frac{\partial^2 N}{\partial z^2} &= 2\zeta, \\ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} &= 0, \end{aligned}\right\} \quad \ldots \ldots \quad (5)$$

The method of integrating these last equations is well known. $L$, $M$, $N$ are the potential functions of imaginary magnetic masses distributed through the space $S_1$ with the densities $\frac{-\xi}{2\pi}, \frac{-\eta}{2\pi}, \frac{-\zeta}{2\pi}$; $P$ is the potential function for masses that lie outside of the region $S$. If we indicate by $r$ the distance from the point $x$, $y$, $z$ to the point whose coördinates are $a$, $b$, $c$; and by $\xi_a$, $\eta_a$, $\zeta_a$ the values of $\xi$, $\eta$, $\zeta$ at the point $a$, $b$, $c$, then

$$\left.\begin{aligned} L &= -\frac{1}{2\pi}\iiint \frac{\xi_a}{r}\,da\,db\,dc, \\ M &= -\frac{1}{2\pi}\iiint \frac{\eta_a}{r}\,da\,db\,dc, \\ N &= -\frac{1}{2\pi}\iiint \frac{\zeta_a}{r}\,da\,db\,dc, \end{aligned}\right\} \quad \ldots \ldots \quad (5a)$$

where the integration is extended over the space $S_1$ and

$$P = \iiint \frac{k}{r}\,da\,db\,dc,$$

where $k$ is an arbitrary function of $a$, $b$, $c$ and the integration is to be extended over the exterior space $S_1$, that includes the region $S$. The

arbitrary function $k$ must be so determined that the boundary conditions are satisfied, a problem whose difficulty is similar to those [difficulties that are met with in problems] on the distribution of electricity and magnetism.

That the values of $u$, $v$, and $w$, given in equation (4), satisfy the condition ($1_4$), is seen at once by differentiation and by considering the fourth of equations (5).

Further, we find by differentiation of equations (4), and considering the first three of equations (5) that:

$$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=2\xi-\frac{\partial}{\partial x}\left[\frac{L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right]$$

$$\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=2\eta-\frac{\partial}{\partial y}\left[\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right]$$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial z}=2\zeta-\frac{\partial}{\partial z}\left[\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}\right]$$

The equations (2) are also equally satisfied when it can be shown that throughout the whole region $S_1$ we have

$$\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

That this is the case results from the equations ($5a$)

$$\frac{\partial L}{\partial x}=\frac{1}{2\pi}\int\int\int\frac{\xi_a(x-a)}{r^3}\,da\,db\,dc,$$

or after partial integration

$$\frac{\partial \mathrm{L}}{\partial x}=\frac{1}{2\pi}\int\int\frac{\xi_a}{r}\,db\,dc-\frac{1}{2\pi}\int\int\int\frac{1}{r}\cdot\frac{\partial\xi_a}{\partial a}\,da\,db\,dc$$

$$\frac{\partial \mathrm{M}}{\partial y}=\frac{1}{2\pi}\int\int\frac{\eta_a}{r}\,da\,dc-\frac{1}{2\pi}\int\int\int\frac{1}{r}\cdot\frac{\partial\eta_a}{\partial b}\,da\,db\,dc$$

$$\frac{\partial \mathrm{N}}{\partial z}=\frac{1}{2\pi}\int\int\frac{\xi_a}{r}da\,db-\frac{1}{2\eta}\int\int\int\frac{1}{r}\cdot\frac{\partial\zeta_a}{\partial c}\,da\,db\,dc.$$

If we add these three equations and again indicate by $d\omega$ the element of the surface of $S$, we obtain:

$$\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}=\frac{1}{2\pi}\int(\xi_a\cos\alpha+\eta_a\cos\beta+\zeta_a\cos\gamma)\frac{1}{r}\,d\omega$$
$$-\frac{1}{2\pi}\int\int\int\frac{1}{r}\left(\frac{\partial\xi_a}{\partial a}+\frac{\partial\eta_a}{\partial b}+\frac{\partial\zeta_a}{\partial c}\right)da\,db\,dc.$$

But since throughout the whole interior of the space $S$ we have

$$\frac{\partial\xi_a}{\partial a}+\frac{\partial\eta_a}{\partial b}+\frac{\partial\zeta_a}{\partial c}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

and since for the whole surface we have

$$\xi_a\cos\alpha+\eta_a\cos\beta+\zeta_a\cos\gamma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

therefore both integrals are equal to zero and the equation ($5b$) as well as the equations (2) are satisfied. The equations (4) and (5) or ($5a$) are thus true integrals of the equations ($1_4$) and (2).

The analogy mentioned in the introduction between the action at a distance of vortex filaments and the electro-magnetic action at a distance of conducting wires, which analogy affords a very good means of making visible the form of the vortex motion, results from this proposition.

When we substitute in the equation (4) the values of $L$, $M$, $N$, from the equation ($5a$) and designate by $\Delta u$, $\Delta v$, $\Delta w$ the infinitely small portions of the velocities $u$, $v$ and $w$ in the integral which depend on the material elements $da$, $db$, $dc$ and designate their resultant by $\Delta p$, we obtain

$$\Delta u = \frac{1}{2\pi} \frac{(y-b)\zeta_a - (z-c)\eta_a}{r^3} da\, db\, dc,$$

$$\Delta v = \frac{1}{2\pi} \frac{(z-c)\xi_a - (x-a)\zeta_a}{r^3} da\, db\, dc,$$

$$\Delta w = \frac{1}{2\pi} \frac{(x-a)\eta_a - (y-b)\xi_a}{r^3} da\, db\, dc.$$

From these equations it follows that,

$$\Delta u(x-a) + \Delta v(y-b) + \Delta w(z-c) = 0,$$

that is to say, $\Delta p$, the resultant of $\Delta u$, $\Delta v$, $\Delta w$, is at right angles to $r$. Further,

$$\xi_a \Delta u + \eta_a \Delta v + \zeta_a \Delta w = 0,$$

that is to say, this same resultant, $\Delta p$, also makes a right angle with the resulting axis of rotation at the point $a$, $b$, $c$. Finally,

$$\Delta p = \sqrt{(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2} = \frac{da\, db\, dc}{2\pi r^2}\ \sigma \sin \nu,$$

where $\sigma$ is the resultant of [the elementary velocities of rotation] $\xi_a$, $\eta_a$, $\zeta_a$, and $\nu$ is the angle between this resultant and $r$, as determined by the equation,

$$\sigma r \cos \nu = (x-a)\xi_a + (y-b)\eta_a + (z-c)\zeta_a$$

*Therefore every rotating particle of liquid* a *causes in every other particle* b *of the same mass of liquid a velocity that is directed perpendicularly to the plane passing through the axis of rotation of the particles* a *and* b. *The magnitude of this velocity is directly proportional to the volume of* a, *to its velocity of rotation, and to the sine of the angle between the line* ab *and the axis of rotation, and inversely proportional to the square of the distance of the two particles.*

The force that an electric current, moving parallel to the axis of rotation at the point $a$, would exert upon a magnetic particle at $b$, follows exactly the same law as above.

The mathematical relationship of both classes of natural phenomena

consists in the fact that in the case of liquid vortices there exists in those parts of the liquid that have no rotation a velocity potential $\varphi$, which satisfies the equation:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0,$$

which equation fails to hold good only within the vortex filaments themselves. But when we imagine the vortex filaments as closed, either within or without the mass of liquid, then the region in which the above differential equation for $\varphi$ holds good is a manifold-connected space, for it remains still connected when we imagine intersecting planes passing through it, each of which is completely bounded by a vortex filament. In such manifold-connected spaces a function $\varphi$ that satisfies the above differential equation becomes many-valued, and it must be many-valued if it is to represent re-entering currents: for since the velocities [$u$, $v$, $w$,] of the liquid particles outside of the vortex filaments are proportional to the [partial] differential coëfficients of $\varphi$ [with reference to $x$, $y$, $z$], therefore, following the liquid particle in its motion one would find the values of $\varphi$ steadily increasing. Therefore, if the current returns into itself, and if one by following it comes finally back to the place where he before was, he will find for this place a second value of $\varphi$ larger than before. Since we can repeat this process indefinitely therefore for every point of such a manifold connected space, there must be an infinite number of different values of $\varphi$, which differ from each other by equal differences, like the different values of

$$\text{tang}^{-1}\left(\frac{x}{y}\right)$$

which is such a many-valued function as satisfies the above differential equation.

The electro-magnetic effects of a closed electric current have relations similar to the preceding. The current acts at a distance as would a certain distribution of magnetic masses over a surface bounded by the conductor. Therefore, outside of such a current the forces that it exerts upon a magnetic particle can be considered as the differential quotients of a potential function $V$ which satisfies the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

Here also the space that surrounds the closed conductor and throughout which this equation holds good, is manifold-connected, and $V$ is many-valued.

Therefore in the vortex motions of liquids, as in the electro-magnetic actions, velocities or forces respectively external to the space occupied by the vortex filaments or the electric currents depend upon many-valued potential functions which moreover satisfy the general differential equations of the magnetic potential function, while on the other hand *within* the space occupied by the vortex filaments or electric cur-

rents, instead of potential functions which can not exist here, there occur other common functions such as are expressed in the equations (4), (5), and (5*a*). On the other hand, for simple progressive movements of liquids and for the magnetic forces, just as for gravitation, for electric attractions and for the steady flow of electricity and heat, we have to do with single-valued potential functions.

The integrals of the hydro-dynamic equations, for which a single-valued velocity potential exists, we can call *integrals of the first class*. Those on the other hand for which there are rotations in one portion of the liquid particles, and correspondingly a many-valued velocity potential for the non-rotating particles we call *integrals of the second class*. It can happen that in the latter case only such portions of the space are to be considered in the problem as contain no rotatory particles of liquid, *e. g.*, in the case of the movements of liquid in a ring-shaped vessel, where a vortex filament can be imagined traversing the axis of the vessel, and where notwithstanding this the problem belongs to those that can be resolved by means of the assumption of a velocity potential.

In the hydro-dynamic integrals of the first class the velocities of the liquid particles have the same direction as, and are proportional to the forces that would be produced by a certain distribution of the magnetic masses outside of the liquid acting on a magnetic particle at the location of the particle of liquid.

In the hydro dynamic integrals of the second class the velocities of the liquid particles have the same direction as, and are proportional to forces acting on the magnetic particle such as would be produced by a closed electric current flowing through the vortex filament and having a density proportional to the velocity of rotation of this filament, combined with the action of magnetic masses entirely outside the liquid. The electric currents within the liquid would flow forward with the respective vortex filaments, and must retain a constant intensity. The adopted distribution of magnetic masses outside of the liquid or on its surface must be so defined that the boundary conditions are satisfied. Every magnetic mass can also, as is well known, be replaced by electric currents. Therefore instead of introducing into the values $u$, $v$, and $w$, the potential function $P$ of an exterior mass $k$, we can obtain an equally general solution if we give to the quantities $\xi$, $\eta$, and $\zeta$ external to the fluid or even only on its surface, such arbitrary values that only closed current filaments arise, and then extend the integration of the equations (5*a*) over the whole region for which $\xi$, $\eta$, and $\zeta$ differ from zero.

## IV. VORTEX SHEETS AND THE ENERGY OF THE VORTEX FILAMENTS.

In the hydro-dynamic integrals of the first class it suffices, as I have already shown, to know the movements of the surface; the movement in the interior is then entirely determined. For the integrals of the second class, on the other hand, the movements of the vortex filaments

located within the fluid are to be determined, taking account of their mutual influences and of the boundary conditions whereby the problem becomes much more complicated. However, for certain simple cases, even this problem can be solved, especially in those cases where the rotations of the liquid particles take place only on certain surfaces or lines and the forms of these surfaces and lines remain unchanged during the translatory motions.

The properties of surfaces that adjoin an indefinitely thin layer of rotating fluid particles are easily seen from the equations (5$a$). When $\xi$, $\eta$, and $\zeta$ differ from zero only within an infinitely thin layer, then, according to well-known propositions, the potential functions L, M, and N will have equal values on both sides of the layer,* but the partial differential coefficients of these functions for the direction normal to the layer will be different on the two sides of the layer. Imagine the coördinate axes so placed that at the point of the vortex sheet under consideration the axis of $z$ corresponds to the normal to the sheet, the axis of $x$ to the axis of rotation of the liquid particles situated in the sheet, so that at this point we have $\eta=\zeta=0$; then will the potentials M and N, as also their partial differential coefficients, have the same values on both sides of the sheet, similarly $L$ and $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial y}$; but $\frac{\partial L}{\partial z}$ will have two different values whose difference is equal to $2\xi\varepsilon$, when $\varepsilon$ indicates the thickness of the stratum. Corresponding to this the equation (4) shows that $u$ and $w$ have the same values on each side of the vortex sheet, but $v$ has values that differ from each other by $2\xi\varepsilon$. Therefore, that component of the velocity that is perpendicular to the vortex line and tangent to the vortex sheet has different values on either side of the vortex sheet. Within the layer of rotating liquid particles we must imagine the respective components of the velocity as uniformly increasing from the value that obtains on one side of the surface to that which obtains on the other side. For when, as here, $\xi$ is constant through the whole thickness of the layer, and we indicate by $\alpha$ a proper fraction, by $v^1$ the value of $v$ on one side, by $v_1$ its value on the other side, by $v_a$ its value within the layer itself at a distance $\alpha\varepsilon$ from the former side; then, as we saw before,

$$v^1-v_1=2\xi\varepsilon$$

because a layer of the thickness $\varepsilon$ and the rotatory velocity $\xi$ lies between the two sides. For the same reasons we must have

$$v^1-v_a=2\xi\varepsilon\alpha=\alpha\,(v^1-v_1),$$

which covers the proposition just enunciated. Since we must think of the rotating liquid particles as themselves moving forward and since the change of distribution on the surface depends on their own motion, therefore we must, through the whole thickness of the layer, attribute

* [This is the "vortex sheet" of English writers.]

to these particles such a mean velocity of progression parallel to the surface as corresponds to the arithmetical mean of the velocities [$v^1$ and $v_1$] prevailing on the two sides of the layer.

For instance such a vortex sheet would be formed when two fluid masses previously separated and in motion come into contact with each other. At the surface of contact the velocities perpendicular thereto must necessarily balance each other. In general the velocities tangent to this surface will, however, be different from each other in the two fluids. Therefore the surface of contact will have the properties of a vortex sheet.

On the other hand, we should not in general think of individual vortex filaments as infinitely slender, because otherwise the velocities on opposite sides of the filament would have infinite values and opposite signs, and therefore the velocity proper of the filament would be indeterminate. In order now to draw certain general conclusions as to the movement of very slender filaments of any sectional area, the principle of the conservation of living force will be made use of.

Therefore before we pass to individual examples, we must first write the equation for the living force K of the moving mass of water, or

$$K=\tfrac{1}{2}h\int\int\int(u^2+v^2+w^2)dx\,dy\,dz. \tag{6}$$

In this integral I substitute from equation (4):

$$u^2=u\left(\frac{\partial P}{\partial x}+\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right)$$

$$v^2=v\left(\frac{\partial P}{\partial y}+\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)$$

$$w^2=w\left(\frac{\partial P}{\partial z}+\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)$$

and integrate by parts; then I indicate by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $\cos\theta$ the angles made by the coördinate axes and the resulting velocity, $q$, respectively with the interior normal to the element $d\omega$ of the mass of liquid and having regard to equations (2) and ($1_4$) I obtain:

$$\begin{aligned}K=-\frac{h}{2}\int d\omega[Pq\cos\theta&+\mathrm{L}(v\cos\gamma-w\cos\beta)\\&+\mathrm{M}(w\cos\alpha-u\cos\gamma)+\mathrm{N}(u\cos\beta-v\cos\alpha)]\\&-h\int\int\int(\mathrm{L}\xi+\mathrm{M}\eta+\mathrm{N}\zeta)dx\,dy\,dz.\end{aligned} \tag{6a}$$

The value of

$$\frac{dK}{dt}$$

is obtained from the equation (1) by multiplying the first by $u$, the second by $v$, the third by $w$, and adding; whence results:

$$h\left(u\frac{du}{dt}+v\frac{dv}{dt}+w\frac{dw}{dt}\right)=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)$$
$$+h\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)$$
$$-\frac{h}{2}\left(u\frac{\partial(q^2)}{\partial x}+v\frac{\partial(q^2)}{\partial y}+w\frac{\partial(q^2)}{\partial z}\right)$$

When we multiply both sides by $dx\,dy\,dz$, then integrate over the whole volume of the liquid mass, and recall that because of $(1_4)$

$$\iiint\left(u\frac{\partial\psi}{\partial x}+v\frac{\partial\psi}{\partial y}+w\frac{\partial\psi}{\partial z}\right)dx\,dy\,dz=-\int\psi\,q\cos\theta\,d\omega,$$

where $\psi$ denotes a function that is continuous and univalued throughout the interior of the liquid mass, we obtain,

$$\frac{dK}{dt}=\int d\omega\,(p-hU+\tfrac{1}{2}hq^2)\,q\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

When the liquid mass is entirely inclosed within rigid walls then at all points of the surface $q\cos\theta$ must be zero, therefore then will $\frac{dK}{dt}=0$, or $K$ become constant.

If we imagine this rigid wall to be at an infinite distance from the origin of coördinates and all vortex filaments that may be present to be at an infinite distance from this origin, then will the potential functions $L$, $M$, $N$ [of imaginary magnetic matter], whose masses $\xi$, $\eta$, $\zeta$, $\left[\text{or densities } \frac{-\xi}{2\pi}, \frac{-\eta}{2\pi}, \frac{-\zeta}{2\pi}\right]$, each and all are equal to zero, diminish at the infinite distance $\mathfrak{R}$ as $\frac{1}{\mathfrak{R}^2}$ and the velocities [which are the partial differential coefficients of $L$, $M$, $N$], will vary as $\frac{1}{\mathfrak{R}^3}$, but the elementary surface $d\omega$, if it is always to correspond to the same solid angle at the origin of the coördinates, will increase as $\mathfrak{R}^2$. The first integral in the expression for $K$, equation $(6a)$, which is extended over the surface of the liquid mass, will diminish as $\frac{1}{\mathfrak{R}^3}$ and therefore will be zero for $\mathfrak{R}$ equal to infinity.

The value of $K$ then reduces to the expression,

$$K=-h\int\int\int(L\xi+M\eta+N\zeta)\,dx\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad (6c)$$

and this quantity is unchanged during the movement.

## V. RECTILINEAR PARALLEL VORTEX FILAMENTS.

We will first investigate the case where only rectilinear vortex threads exist parallel to the axis of $z$, either within a liquid mass of infinite extent or which comes to the same thing, in one that is bounded by two infinite planes perpendicular to the vortex filaments. In this case

all motions take place in planes that are perpendicular to the axis of $z$ and are precisely the same in all such planes.

Therefore we put

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial p}{\partial z} = \frac{\partial V}{\partial z} = 0.$$

Then equations (2) reduce to

$$\xi = 0,\ \eta = 0,\ 2\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x},$$

the equations (3) become

$$\frac{\delta \zeta}{\delta t} = 0$$

Therefore the vortex threads, in so far as they have constant sectional areas, have also constant velocities of rotation.

The equations (4) reduce to,

$$u = \frac{\partial N}{\partial y}, \quad v = \frac{\partial N}{\partial x}, \quad \frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} = 2\zeta.$$

In this I have put $P = 0$ in accord with the remark in Sect. III. Therefore the equation of the streamline is $N =$ constant.

In this case $N$ is the potential function of infinitely long lines; this function itself is infinitely large, but its differential coefficients are finite. Let $a$ and $b$ be the coördinates of a vortex filament the area of whose cross-section is $da\, db$, then is

$$-v = \frac{\partial N}{\partial x} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{x-a}{r^2},$$

$$u = \frac{\partial N}{\partial y} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{y-b}{r^2}.$$

Hence it follows that the resultant velocity $q$ is perpendicular to the $r$ drawn perpendicular to the vortex filament and its value is

$$q = \frac{\zeta\, da\, db}{\pi r}.$$

If within a liquid mass of indefinite extent in the direction $x$ and $y$ we have many vortex filaments whose coördinates are respectively $x_1, y_1$; $x_2, y_2$, etc., while the products of rotatory velocity by the sectional area are for each distinguished by $m_1$, $m_2$, etc., and if we form the sums,

$$U = m_1\, u_1 + m_2\, u_2 + m_3\, u_3, \text{ etc.},$$

$$V = m_1\, v_1 + m_2\, v_2 + m_3\, v_3, \text{ etc.},$$

then these sums are each equal to zero, because that part of each sum that is due to the action of the second vortex filament on the first is counterbalanced by the action of the first vortex filament on the second. That is to say, the two effects are, respectively,

$$m_1 \cdot \frac{m_2}{\pi} \cdot \frac{x_1 - x_2}{r^2} \text{ and } m_2 \cdot \frac{m_1}{\pi} \cdot \frac{x_2 - x_1}{r^2},$$

and so on through all the other pairs of sums. Now $U$ is the velocity in the direction of $x$, of the center of gravity of the masses $m_1$, $m_2$, etc., multiplied by the sum of these masses; similarly $V$ is the velocity taken in the direction of $y$. Both velocities are therefore zero, unless the sum of the masses is zero, in which case there is no center of gravity at all. Therefore the center of gravity of the vortex filaments remains unchanged during their motion, and since this proposition holds good for every distribution of the vortex filaments, therefore we may also apply it to the individual filaments of infinitely small cross section.

Hence result the following consequences:

(1) If we have but one individual rectilinear vortex filament of infinitely small cross-section within a liquid mass of infinite extent in all directions perpendicular to the filament, then the movement of the particles of water at a finite distance from the filament depends only on the product $\xi\, da\, db = m$, or the velocity of rotation multiplied by the area of the cross-section, and not on the form of the cross-section. The liquid particles rotate about the filament with the tangential velocity $\frac{m}{\pi r}$ where $r$ denotes the distance from the center of gravity of the vortex filament. The location of the center of gravity, the velocity of rotation, the area of the cross section, and therefore also the quantity $m$ remains unchanged although the form of the infinitely small cross-section may change.

(2) If we have two rectilinear vortex filaments of infinitely small cross-sections and an indefinitely large liquid mass, each will drive the other in a direction that is perpendicular to the line joining them together. The length of this connecting line will not be changed thereby; therefore both will revolve about their common center of gravity, remaining at equal constant distances therefrom. If the rotatory velocity is in the same direction in the two filaments and therefore has the same sign, then their center of gravity must lie between them. If the rotations are mutually opposed to each other and therefore of opposite signs, then their center of gravity lies in the prolongation of the line connecting the filaments. If the products of the rotatory velocity by the cross section are numerically equal for the two but of opposite signs, thereby causing the center of gravity to be at an infinite distance, then both filaments advance with equal velocity and in the same direction perpendicular to their connecting line.

The case where a vortex filament of infinitely small section lies close to an infinitely extended plane surface parallel to it can be reduced to this last case. The boundary condition for the movement of the liquid along a plane (*i. e.*, that the motion must be parallel to this plane) is satisfied when we imagine a second vortex filament, which is as the reflected image of the first, introduced on the other side of the plane. Hence it follows that the vortex filament within the liquid mass ad-

vances parallel to the plane in the direction in which the liquid particles, between it and the plane, themselves move, and with one-fourth of the velocity possessed by the particles that are at the foot of the perpendicular drawn from the filament to the plane.

The assumption of the infinitely small cross-section leads to no inadmissible results, because each individual filament exerts no force upon itself affecting its own progression, but is driven forwards only by the influence of the other filaments that may be present [or by the action at the boundary]. But it is otherwise in the case of curved filaments.

## VI. CIRCULAR VORTEX FILAMENTS.

In a liquid mass of indefinite extent let there be present only circular filaments whose planes are perpendicular to the axis of $z$, and whose centers lie in this axis, so that all are symmetrical about this axis. Transform the coördinates by putting

$$\begin{aligned} x &= \chi \cos \varepsilon, & a &= g \cos e, \\ y &= \chi \sin \varepsilon, & b &= g \sin e, \\ z &= z, & c &= c. \end{aligned}$$

Agreeably to the assumption just made, the velocity of rotation $\sigma$ is only a function of $\chi$ and $z$, or of $g$ and $c$, and the axis of rotation is everywhere perpendicular to $\chi$ (or $g$) and to the axis of $z$. Therefore the rectangular components of the rotation at this point whose coördinates are $g$, $e$, and $c$ become

$$\xi = -\sigma \sin e, \quad \eta = \sigma \cos e, \quad \zeta = 0.$$

In the equation (5$a$) we now have,

$$r^2 = (z-c)^2 + \chi^2 + g^2 - 2\chi g \cos(\varepsilon - e)$$

$$L = \frac{1}{2\pi} \int\int\int \frac{\sigma \sin e}{r} g\, dg\, de\, dc$$

$$M = -\frac{1}{2\pi} \int\int\int \frac{\sigma \cos e}{r} g\, dg\, de\, de$$

$$N = 0$$

From the equations for $L$ and $M$ by multiplying by $\cos \varepsilon$ and $\sin \varepsilon$ and adding we obtain

$$L \sin \varepsilon - M \cos \varepsilon = -\frac{1}{2\pi} \int\int\int \frac{\sigma \cos(\varepsilon - e)}{r} g\, dg\, d(\varepsilon - e)\, de,$$

$$L \cos \varepsilon + M \sin \varepsilon = \frac{1}{2\pi} \int\int\int \frac{\sigma \sin(\varepsilon - e)}{r} g\, dg\, d(\varepsilon - e)\, de,$$

In both these integrals the angles $e$ and $\varepsilon$ occur only in the connection $(\varepsilon - e)$ and this quantity can therefore be considered as the variable under the sign of integration. In the second integral the terms that contain $(\varepsilon - e) = \mathfrak{c}$ balance those that contain $(\varepsilon - e) = 2\pi - \mathfrak{c}$; therefore this integral is equal to zero.

Therefore if we put

$$\psi = \frac{1}{2\pi} \int\int\int \frac{\sigma \cos e\, g\, dg\, de\, dc}{\sqrt{(z-c)^2 + \chi^2 + g^2 - 2g\chi \cos e}} \tag{7}$$

then will

$$M \cos \varepsilon - L \sin \varepsilon = \psi$$
$$M \sin \varepsilon + L \cos \varepsilon = 0,$$

or

$$L = -\psi \sin \varepsilon, \quad M = \psi \cos \varepsilon. \tag{7a}$$

Let $\tau$ denote the velocity in the direction of the radius $\chi$, and consider the fact that on account of the symmetrical position of the vortex ring in reference to the axis $z$ the velocity must be zero in the direction of the circumference of the circle, we must have

$$u = \tau \cos \varepsilon, \; v = \tau \sin \varepsilon$$

and according to equations (4)

$$u = \frac{\partial M}{\partial z}, \quad v = \frac{\partial L}{\partial z}, \quad w = \frac{\partial L}{\partial x} - \frac{\partial M}{\partial y}.$$

Hence it follows that

$$\tau = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial \chi} + \frac{\psi}{\chi}$$

or

$$\tau \chi = -\frac{\partial (\psi \chi)}{\partial z}, \quad w\chi = \frac{\partial (\psi \chi)}{\partial \chi}. \tag{7b}$$

Therefore the equation of the stream line is

$$\psi \chi = \text{const.}$$

When we execute the integrations indicated in the value of $\psi$, first for a vortex filament of infinitely small cross-section, putting therein $m_1 = \sigma\, dg\, dc$ and indicating by $\psi_{m_1}$ the part of $\psi$ depending thereon, we have

$$\psi_{m_1} = \frac{m_1}{\pi} \sqrt{\frac{g}{\chi}} \left\{ \frac{2}{\varkappa}(F-E) - \varkappa F \right\},$$

$$\varkappa^2 = \frac{4g\chi}{(g+\chi)^2 + (z-c)^2},$$

wherein $F$ and $E$ indicate the complete elliptic integrals of the first and second order respectively for the modulus $\varkappa$.

For brevity we put

$$U = \frac{2}{\varkappa}(F-E) - \varkappa F,$$

where $U$ is therefore a function of $\varkappa$, then is

$$\tau\chi = \frac{m_1}{\pi} \sqrt{g\chi} \cdot \frac{\partial U}{\partial \varkappa} \cdot \varkappa \cdot \frac{z-c}{(g-\chi)^2 + (z-c)^2}.$$

If now a second vortex filament $m$ exist at the point determined by $\chi$ and $z$, and if we let $\tau_1$ be the velocity in the direction of $g$ that $m$ communicates to the filament $m_1$, we then obtain the value of this ve-

locity if in the expression for $\tau$ we substitute $\tau_1$, $g$, $\chi$, $c$, $z$, $m$, in place of $\tau$, $\chi$, $g$, $z$, $c$, $m_1$.

In this process $\varkappa$ and $U$ remain unchanged and we obtain,

$$m\tau\chi+m_1\tau_1 g=0. \qquad (8)$$

If now we determine the value of the velocity $w$ parallel to the axis, caused by the vortex filament $m_1$ whose coördinates are $g$ and $c$, we find:

$$w\chi=\tfrac{1}{2}\frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\,U+\frac{m_1}{\pi}\sqrt{g\chi}\frac{\partial U}{\partial\varkappa}\cdot\frac{\varkappa}{2\chi}\cdot\frac{(z-c)^2+g^2-\chi^2}{(g+\chi)^2+(z-c)^2}.$$

If now we call $w_1$ the velocity at the locality of $m_1$ parallel to the axis of $z$, which is caused by the vortex ring $m$ whose coördinates are $z$ and $\chi$, then in order to determine this, we only need to execute the interchange of appropriate coördinates and masses as above shown. Thus we find:

$$2mw\chi^2-2\,m_1\,w_1\,g^2-m\tau\chi z-m_1\tau_1 gc=\frac{2mm_1}{\pi}\sqrt{g\chi}\,U. \qquad (8a)$$

Sums similar to (8) and (8$a$) can be found for any number of vortex rings. For the $n$th of these rings I designate the product $\sigma\,dg\,dc$ by $m_n$; the components of the velocity that is communicated to this ring by all the other rings are $\tau_n$ and $w_n$, in which however I provisionally omit the velocities that every vortex ring can communicate to itself. Further I call the radius of this ring $\rho_n$ and its distance from a surface perpendicular to the axis $\lambda$, which two latter quantities agree with $\chi$ and $z$ as to direction, but, as belonging to this particular ring, they are functions of the time and not independent variables as are $\chi$ and $z$. Finally let the value of $\psi$, in so far as it depends on the other vortex rings, be $\psi_n$. By forming and adding the equations (8) and (8$a$) corresponding to each pair of vortex rings, there results

$$\Sigma\,[m_n\,\rho_n\,\tau_n]=0.$$

$$\Sigma\,[2\,m_n\,w_n\,\rho^2_n-m_n\,\tau_n\,\rho_n\,\lambda_n]=\Sigma\,[m_n\,\rho_v\,\psi_n].$$

So long as we have in these sums only a finite number of separate and infinitely slender vortex rings, we must understand by $w$, $\tau$, and $\psi$ only those parts of these quantities that are due to the presence of the other rings. But when we imagine an infinite number of such rings keeping the space continuously filled, then $\psi$ becomes the potential function of a continuous mass, $w$ and $\tau$ become partial differential coefficients of this potential function, and it is known* that both for such functions and for their differential coefficients, the portions of the function that depend upon the presence of matter within an infinitely small space surrounding a point for which the function is determined are infinitely small with respect to those portions that depend on finite masses at finite distances.

---

* See Gauss, *Allgemeine Theorie des Erdmagnetismus* in the *Resultate des magnetischen Vereins im Jahre*, 1839, page 7, or the translation in Taylor's Scientific Memoirs, vol. II.

Therefore if we change the sums into integrals we can understand by $w$, $\tau$, and $\psi$ the total values of these quantities that exist at the point in question, and can put

$$w=\frac{d\lambda}{dt}, \qquad \tau=\frac{d\rho}{dt}.$$

To this end we replace the quantity $m$ by the product $\sigma d\rho d\lambda$, and the summations thus become converted into the following integrals:

$$\int\int\sigma\rho\frac{d\rho}{dt}d\rho\, d\lambda=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$2\int\int\sigma\rho^2\frac{d\lambda}{dt}d\rho\, d\lambda-\int\int\sigma\rho\lambda\frac{d\rho}{dt}d\rho\, d\lambda=\int\int\sigma\rho\psi\, d\rho\, d\lambda \quad . \quad . \quad (9a)$$

Since, in accordance with Sect. II, the product $\sigma\, d\rho\, d\lambda$ does not vary with the time, therefore, the equation (9) can be integrated with respect to $t$, and we obtain

$$\tfrac{1}{2}\int\int\sigma\rho^2\, d\rho\, d\lambda=\text{Const.}$$

Imagine the space divided by a plane that passes through the axis of $z$, and therefore intersects all the vortex rings that are present; then consider $\sigma$ as the density of one layer of the mass, and let $\mathfrak{M}$ be the total mass in this layer adjoining this dividing plane; therefore,

$$\mathfrak{M}=\int\int\sigma\, d\rho\, d\lambda,$$

and let $R^2$ be the mean value of $\rho^2$ for all the elementary masses, then

$$\int\int\sigma\rho.\,\rho d\rho\, d\lambda=\mathfrak{M}\, R^2,$$

and, since this integral and the value of $\mathfrak{M}$ do not vary with the time, it follows that $R$ also remains unchanged during the motion of translation.

Therefore if there exists in the unlimited mass of liquid only one circular vortex filament of infinitely small sectional area, then its radius remains unchanged.

According to equation (6$c$), the total living force in our case is

$$\begin{aligned} K&=-h\int\int\int(L\xi+M\eta)da\, db\, dc. \\ &=-h\int\int\int\psi\sigma\cdot\rho\, d\rho\, d\lambda\, d\varepsilon. \\ &=-2\pi h\int\int\psi\sigma\cdot\rho\, d\rho\, d\lambda. \end{aligned}$$

This also does not change with time.

Furthermore, because $\sigma\, d\rho\, d\lambda$ does not vary with time, therefore,

$$\frac{d}{dt}\int\int\sigma\rho^2\lambda\, d\rho\, d\lambda=2\int\int\sigma\rho\lambda\frac{d\rho}{dt}d\rho\, d\lambda+\int\int\sigma\rho^2\frac{d\lambda}{dt}d\lambda\, d\rho;$$

therefore if we indicate by $l$ the value of $\lambda$ for the center of gravity of the vortex filament treated of in equation (9$a$), and multiply (9) by this $l$, and add the result to (9$a$), and substitute therein the equation last given, we obtain

$$2\frac{d}{dt}\int\int\sigma\rho^2\lambda\, d\rho\, d\lambda+5\int\int\sigma\rho(l-\lambda)\frac{d\rho}{dt}d\rho\, d\lambda=-\frac{K}{2\pi h} \quad . \quad . \quad (9b)$$

When the section of the vortex thread is infinitely small and $\varepsilon$ is an infinitely small quantity of the same order as $(l-\lambda)$ and the remaining linear dimensions of the section, but $\sigma\, d\rho\, d\lambda$ is finite, then $\psi$ and also $K$ are of the same order of infinitely large quantities as log $\varepsilon$. For very small values of the distance $v$ from the vortex ring we have

$$v=\sqrt{(g-\chi)^2+(z-c)^2},$$

$$\varkappa^2=1-\frac{v^2}{4\, g^2},$$

$$\psi\, m_1=\frac{m_1}{\pi}\log\left(\sqrt{\frac{1-\varkappa^2}{4}}\right)=\frac{m_1}{\pi}\log\frac{v}{8g}.$$

In the value of $K$, $\psi$ is multiplied by $\rho$ or $g$. If $g$ is finite, and $v$ of the same order as $\varepsilon$, then $K$ is of the same order as log $\varepsilon$. Only when $g$ is infinitely large of the order $\frac{1}{\varepsilon}$ will $K$ be infinitely large of the order $\frac{1}{(\varepsilon)}\log\varepsilon$. But in this case the circle becomes a straight line. On the other hand, if $\frac{d\rho}{dt}$, which is equal to $\frac{\partial \psi}{\partial z}$, is of the order $\frac{1}{\varepsilon}$, then the second integral will be finite, and for a finite value of $\rho$ will be infinitely small with respect to $K$. In this case we can, in the first integral, substitute the constant $l$ in place of $\lambda$ and obtain

$$2\frac{d(\mathfrak{M}R^2l)}{dt}=-\frac{K}{2\pi h}$$

or

$$2\mathfrak{M}R^2l=C-\frac{K}{2\pi h}t$$

Since $\mathfrak{M}$ and $R$ are constant, $l$ can only vary proportionally to the time. When $\mathfrak{M}$ is positive the motion of the liquid particles on the outer side of the ring is directed toward the side of positive $z$, but on the inner side of the ring toward the negative $z$. $K$, $h$, and $R$ are by their nature always positive.

Hence it follows that for a circular vortex filament of very small cross-section in an infinitely extended mass of liquid the center of gravity of a cross-section has a motion parallel to the axis of the vortex ring, which is of approximately constant and very large velocity, and which is directed toward the same side as that toward which the liquid flows through the ring. Infinitely slender vortex filaments of a finite radius will have infinitely large velocities of propagation. But if the radius of the vortex ring is infinitely large of the order $\frac{1}{(\varepsilon)}$, then will $R^2$ be infinitely large with respect to $K$, and $l$ will be constant. The vortex filament which has thus transformed itself into a straight line will be stationary, as we had already previously found for rectilinear vortex filaments.

We can now in general see how two circular vortex threads having a common axis will behave with respect to each other, since each one independent of its own translatory motion also follows the movement of the liquid particles caused by the other filament. If they have the same direction of rotation, then they both advance in the same direction, and at first the preceding one enlarges, then it advances more slowly while the following one diminishes and advances more rapidly; finally, if the progressive velocities are not too different, the second catches up with the first and passes through it. Then the same performance is repeated by the one that is now in the rear so that the rings alternately pass through each other.

If the vortex filaments have the same radii, but equal and opposite rotatory velocities, then they will approach each other and simultaneously enlarge, so that finally when they have come very close together their movement towards each other grows continually feebler, while on the other hand the enlargement goes on with increasing rapidity. If the two vortex threads are perfectly symmetrical, then midway between them the velocity of the liquid particles in the direction parallel to the axis is equal to zero. Therefore one can imagine a rigid wall located here without disturbing the motion and thus obtain the case of a vortex ring that encounters a rigid wall.

I remark further that we can easily study these movements of circular vortex rings in nature if we draw a half-immersed circular disk or the approximately semicircular end of a spoon rapidly for a short distance along the surface of a liquid and then quickly draw it out. There then remain in the liquid semi-vortex rings whose axes lie in the free upper surface of the liquid. The free upper surface thus forms, for the liquid mass, a boundary plane that passes through the axis whereby no important change is made in the motions. The vortex rings advance, broaden when they encounter a screen, and are enlarged or diminished by the action of other vortex rings precisely as we have deduced from the theory.

## III.

## ON DISCONTINUOUS MOTIONS IN LIQUIDS.*

By Prof. H. von Helmholtz.

It is well known that the hydro-dynamic equations give precisely the same partial differential equations for the interior of an incompressible fluid that is not subject to friction and whose particles have no motion of rotation, as obtain for stationary currents of electricity or heat in conductors of uniform conductivity. One might therefore expect that for the same external form of the space traversed by the current and for the same boundary conditions the form of the current (except for differences depending on small incidental conditions), would be the same for liquids, for electricity, and for heat. In reality however in many cases there exist easily recognizable and very fundamental differences between the currents in a liquid and the above mentioned imponderables.

Such differences are especially notable when the currents flowing through an opening with sharp edges enter into a wider space. In such cases the stream lines of electricity radiate from the opening outwards immediately towards all directions, while a flowing fluid, water as well as air, moves from the opening at first forward in a compact stream which at a less or greater distance then ordinarily resolves itself into a whirl. The portions of the fluid in the larger receiving vessel lying near the opening but outside the stream can, on the other hand, remain almost at perfect rest. Every one is familiar with this mode of motion, especially as a current of air impregnated with smoke shows it very plainly. In fact the compressibility of the air does not come much into consideration in these processes, and with slight variations air shows the same forms of motion as does water.

On account of the great differences between the facts as observed and the results of theoretical analysis as hitherto achieved the hydro-dynamic equations must necessarily appear to the physicist as a prac-

* From the *Monatsberichte* of the Royal Academy of Science, Berlin. 1868, April 23, pp. 215–228. Helmholtz *Wissenschaftliche Abhandlungen*, vol. I, pp. 146–157. Berlin, 1882.

tically very imperfect approximation to the reality. The cause of this might be suspected to lie in the internal friction or viscosity of the fluid, although all forms of infreqent and sudden irregularities (with which certainly everyone has to contend who has instituted observations on the movements of fluids) can evidently never be explained as the effect of the steadily and uniformly acting friction.

The investigation of cases where periodical movements are excited by a continuous current of air, as, for example, in organ pipes, showed me that such an effect could only be produced by a discontinuous motion of the air, or at least by a kind of motion coming very near to it, and this has lead me to the discovery of a condition that must be taken into consideration in the integration of the hydro-dynamic equations, and that, so far as I know, has been overlooked hitherto, whose consideration on the other hand, in those cases where the computation can be carried out, really gives, in fact, forms of motions such as those that are actually observed. This condition is due to the following circumstance:

In the hydro-dynamic equations the velocity and the pressure of the flowing particles are treated as continuous functions of the coördinates. On the other hand, there is no reason in the nature of a liquid, if we consider it as perfectly fluid, therefore not subject to viscosity, why two contiguous layers of liquid should not glide past each other with definite velocities. At least those properties of fluids that are considered in the hydro-dynamic equations, namely, the constancy of the mass in each element of space and the uniformity of pressure in all directions, evidently furnish no reasons why tangential velocities of finite difference in magnitude should not exist on both sides of a surface located in the interior. On the other hand, the components of velocity and of pressure perpendicular to the surface must of course be equal on both sides of such a surface. I have already in my memoir on vortex motions called attention to the fact that such a case must occur when two moving masses of liquid previously separate and having different motions come to have their surfaces in contact. In that memoir I was led to the idea of such a surface of separation,* or vortex surface as I there called it† through the fact that I imagined a system of parallel vortex filaments arranged continuously over the surface whose mass was indefinitely small without losing their moment of rotation.

Now, in a liquid at first quiet or in continuous motion a definite difference in the movement of immediately adjoining particles of liquid can only be brought about through moving forces acting discontinuously. Among the external forces the only one that can here come into consideration is impact.

But in the interior of liquids there is also a cause present that can

[* Ordinarily called *surface of discontinuity* or "a discontinuous surface" by English writers.]

[† That is, an infinitely thin layer of parallel vortex filaments, the "*vortex sheet*" of English writers.]

bring about discontinuity of motion—namely, the pressure, which can assume any positive value whatever while the density of the liquid will continuously vary therewith; but as soon as the pressure passes the *zero* value and becomes negative, a discontinuous variation of the density occurs; the liquid is torn asunder.

Now, the magnitude of the pressure (at any point) in a moving fluid depends on the velocity (at that point), and in incompressible fluids the diminution of pressure under otherwise similar circumstances is directly proportional to the living force of the moving particles of liquid. Therefore if the latter exceeds a certain limit the pressure must, in fact, become negative, and the liquid tears asunder. At such a place the accelerating force, which is proportional to the differential quotient of the pressure, is evidently discontinuous, and thus the condition is fulfilled which is necessary in order to bring about a discontinuous motion of the liquid. The movement of the liquid past any such place can now take place only by the formation from that point onward of a surface of discontinuity.

The velocity that will cause the tearing asunder of the liquid is that which the liquid would assume when it flows into empty space under the pressure that the liquid would have at rest at the point in question. This is indeed a relatively considerable velocity; but it is to be remarked that if liquids flow continuously like electricity the velocity at every sharp edge around which the current bends must be infinitely great.* Thence it follows that *at every geometrically perfect sharp edge past which liquids flow, even for the most moderate velocity of the rest of the liquid, it must be torn asunder and form a surface of discontinuity.* On the other hand, for imperfectly somewhat rounded edges such phenomena first occur for certain larger velocities. Pointed protuberances on the surface of a canal through which a current flows will have similar effects.

As concerns gases, the same circumstance occurs as with liquids, only with this difference,—that the living force of the motion of a particle is not directly proportional to the diminution of the pressure ($p$); but taking into consideration the cooling of the air by its expansion the living force is proportional to the diminution of $p^m$, where $m = 1 - \frac{1}{\gamma}$ and $\gamma$ is the ratio of the specific heat at constant pressure to that for constant volume. For atmospheric air the exponent $m$ has the value 0.291. Since this is positive and real, therefore $p^m$, like $p$, for high values of the velocity can only diminish to zero and not become negative. It would be otherwise if gases simply followed the law of Mariotte and experienced no change of temperature. Then instead of $p^m$ the quantity $\log p$ would occur, which can become negative and infinite without

*At the very small distance $\rho$ from a sharp edge whose surfaces meet each other at the angle $\alpha$ the velocities will be infinite, or as $\rho^{-m}$, where $m = \frac{\pi - \alpha}{2\pi - \alpha}$.

$p$ being negative. Under this condition the tearing asunder of the mass of air would not be necessary.

It is possible to convince one's self of the actual existence of such discontinuities when we allow a stream of air impregnated with smoke to issue from a round opening or a cylindrical tube with moderate velocity so that no hissing occurs. Under favorable circumstances one obtains thin rays or jets of this kind of a few lines diameter and a length of many feet. Within the cylindrical surface the air is in motion with constant velocity, but outside it, on the other hand, in the immediate neighborhood of the jet it moves not at all or very slightly. One sees this very sharp separation clearly when we conduct a steadily flowing cylindrical jet of air through the point of a flame, out of which it cuts a sharply defined piece, while the rest of the flame remains entirely undisturbed, and at most a very thin stratum of flame, which corresponds to the boundary layer of the jet influenced by friction, is carried along a little way.

As concerns the mathematical theory of this motion I have already given the boundary conditions for the existence of an interior surface of separation within the liquid. They consist in this that the pressures on both sides the surface must be equal and equally so the components of the velocity normal to the discontinuous surface. Since now the movement throughout the entire interior of a liquid whose particles have no motion of rotation is wholly determined when the motion of its entire exterior surface and its interior discontinuities are given, therefore in general for a liquid whose exterior boundary is fixed, it is only necessary to know the movement of the surfaces of separation and the variations of the discontinuity.

Now such a discontinuous surface can be treated mathematically precisely as if it were a *vortex sheet*, that is to say, as if it were continuously enveloped by vortex filaments of indefinitely small mass but finite moments of rotation. For each element of such a vortex sheet there is a direction for which the components of the tangential velocities are equal. This gives at once the direction of the vortex filaments at the corresponding place. The moment of this filament is to be put proportional to the difference existing between the components, taken perpendicular to it, of the tangential velocity on both sides of the surface.

The existence of such vortex filaments in an ideal frictionless liquid is a mathematical fiction that facilitates the integration. In a real liquid subject to friction, this fiction becomes at once a reality inasmuch as by the friction the boundary particles are set in rotation, and thus vortex filaments originate there having finite gradually increasing masses, while the discontinuity of the motion is thereby at the same time compensated.

The motion of a vortex sheet and the vortex filaments lying in it is to be determined by the rules established in my *Memoir on Vortex*

*Motions.* The mathematical difficulties of this problem however can be overcome only in a few of the simpler cases. In many other cases, however, one can from the above given method of consideration of this matter at least draw conclusions as to the general nature of the variations that occur.

Especially is it to be mentioned that in accordance with the laws established for vortex motions, the vortex filaments and with them the vortex sheets in the interior of a frictionless liquid can neither originate nor disappear, but rather each vortex filament must retain permanently the same constant moment of rotation; furthermore that the vortex filaments themselves advance along the vortex sheet with a velocity that is the mean of the two velocities existing on the two sides of the discontinuous surface. Thence it follows that *a surface of discontinuity can only elongate in the direction towards which the stronger* of the two currents that meet in it is directed.

I have first sought to find examples of permanent discontinuous surfaces in steady currents, for which the integration can be executed, in order thereby to prove whether the theory gives forms of currents that correspond to experience better than when we disregard the discontinuity of motion. If a surface of discontinuity that separates quiet and moving water from each other is to remain stationary, then along this surface the pressure within the moving layer must be the same as in the quiet layer, whence it follows that the tangential velocity of the particles of liquid must be constant throughout the whole extent of the surface; equally so must the density of the fictitious vortex filament be constant. The beginning and end of such a surface can only lie on the boundary of the inclosure or at infinity. Where the former alternative is the case they must be tangent to the wall of the inclosure, assuming that the latter is continuously curved, because the component-velocity normal to the wall of the inclosure must be zero.

Moreover the stationary forms of the discontinuous surface are distinguished, as experiment and theory agree in showing, by a remarkably high degree of variability under the slightest perturbations, so that to a certain extent they behave similarly to bodies in unstable equilibrium. The astonishing sensitiveness to sound waves of a cylindrical jet of air impregnated with smoke has already been described by Tyndall; I have confirmed this observation. This is evidently a peculiarity of surfaces of discontinuity that is of the greatest importance in operating sonorous pipes.

Theory allows us to recognize that in general wherever an irregularity is formed on the surface of an otherwise stationary jet, this must lead to a progressive spiral unrolling of the corresponding portion of the surface, which portion, moreover, slides along the jet. This tendency towards spiral unrolling at every disturbance is moreover easy to see in the observed jets. According to the theory a prismatic or cylindrical jet can be indefinitely long. In fact however such an one can not be

formed, because in an element so easily moved as is the air small disturbances can never be entirely avoided.

It is easy to see that such an endless cylindrical jet, issuing from a tube of corresponding section into a quiet exterior fluid and everywhere containing fluid that is moving with uniform velocity parallel to its axis, corresponds to the requirements of the "steady condition."

I will here further sketch only the mathematical treatment of a case of the opposite kind, where the current from a wide space flows into a narrow canal, in order thereby also at the same time to give an example of a method by which some problems in the theory of potential functions can be solved that hitherto have been attended by difficulties.

I confine myself to the case where the motion is steady and dependent only upon two rectangular coördinates, $x$ and $y$; where moreover no rotating particles are present in the frictionless fluid at the beginning, and where none such can be subsequently formed. If we indicate by $u$ the component parallel to $x$ of the velocity of the fluid particle at the point $(xy)$ and by $v$ the velocity parallel to $y$, then, as is well known, two functions of $x$ and $y$ can be found such that

$$\left.\begin{aligned} u &= \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By these equations the conditions are also directly fulfilled that in the interior of the fluid the mass shall remain constant in each element of space, viz:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

For a constant density, $h$, and when the potential of the external forces is indicated by $v$, the pressure in the interior is given by the equation—

$$v - \frac{p}{h} + c = \tfrac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] = \tfrac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \quad . \quad . \quad . \quad . \quad (1b)$$

The curves

$$\psi = \text{constant}$$

are the stream lines of the fluid, and the curves

$$\varphi = \text{constant}$$

are orthogonal to them. The latter are the equi-potential curves when electricity, or the equal temperature curves when heat, flows in steady currents in conductors of uniform conductivity.

From the equation (1) it follows as an integral equation that the quantity $\varphi + \psi i$ is a function of $x + yi$, where $i = \sqrt{-1}$. The solutions hitherto found generally express $\varphi$ and $\psi$ as the sums of terms that are

themselves functions of $x$ and $y$. But inversely we can consider and develop $x + yi$ as a function of $\varphi + \psi i$. In problems relative to currents between two stationary walls, $\psi$ is constant along the boundaries, and therefore if $\varphi$ and $\psi$ are presented as rectangular coördinates in a plane, then in a strip of this plane bounded by two parallel straight lines, $\psi = c_0$ and $\psi = c_1$, the function $x + yi$ is to be so taken that on the edge it corresponds to the equation of the wall, but in the interior it assumes a given variability.

A case of this kind occurs when we put

$$x + yi = A\left\{\varphi + \psi + e^{\phi + \psi i}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$x = A\varphi + Ae^{\phi}\cos\psi$$
$$y = A\psi + Ae^{\phi}\sin\psi$$

for the value $\psi = \pm\pi$ we have $y$ constant and $x = A\varphi - Ae^{\phi}$.

When $\varphi$ varies from $-\infty$ to $+\infty$ the value of $x$ changes at the same time from $-\infty$ to $-A$, and then again back to $-\infty$.

The stream lines $x = \pm\pi$ correspond thus to a current along two straight walls, for which $y = \pm A\pi$ and $x$ varies between $-\infty$ and $-A$.

Therefore when we consider $\psi$ as the expression of the stream curve the equation (2) corresponds to the flow out into endless space from a canal bounded by two parallel planes. On the border of the canal however where $x = -A$ and $y = \pm A\pi$ and where further, $\varphi = 0$ and $\psi = \pm\pi$, we have

$$\left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 = 0,$$

therefore

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \infty$$

Electricity and heat flow in this manner, but liquids must tear asunder.

If from the border of the canal there extend stationary dividing discontinuous lines that are of course prolongations of the stream lines $\psi = \pm\pi$ that follow along the wall and if outside of these discontinuous lines that limit the flowing fluid there is perfect quiet, then must the pressure be the same on both sides of these dividing lines. That is to say, along that portion of the line $\psi = \pm\pi$ which corresponds to the free dividing line, in accordance with the equation (1*b*), we must have

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In order now, in the solution of this modified problem, to retain the fundamental idea of the motion expressed in equation (2), we will add

to the above expression of $x+y\,i$ still another term $\sigma+\tau\,i$, which is also always a function of $\varphi+\psi\,i$, we have then

$$\left.\begin{aligned} x &= A\,\varphi + A\,e^{\phi}\cos\psi + \sigma \\ y &= A\,\psi + A\,e^{\phi}\sin\psi + \tau \end{aligned}\right\} \qquad (3a)$$

and must determine $\sigma+\tau i$ so that along the free portion of the discontinuous surface where $\psi = \pm\pi$ we shall have

$$\left(A - Ae^{\phi} + \frac{\partial\sigma}{\partial\varphi}\right)^2 + \left(\frac{\partial\tau}{\partial\varphi}\right)^2 = \text{constant.}$$

This condition is fulfilled if we make

$$\frac{\partial\sigma}{\partial\varphi} = 0 \text{ or } \sigma = \text{Constant} \qquad (3b)$$

and

$$\frac{\partial\tau}{\partial\varphi} = \pm A\sqrt{2e^{\phi} - e^{2\phi}} \qquad (3c)$$

Since $\psi$ is constant along the wall we can integrate the last equation with respect to $\varphi$, and change the integral into a function of $\varphi+\psi\,i$ by substituting everywhere instead of $\varphi$ the expression $\varphi+i\,(\psi+\pi)$. Thus by an appropriate determination of the constants of integration we obtain

$$\sigma+\tau\,i = Ai\left\{\sqrt{-2e^{(\phi+\psi\,i)} - e^{(2\phi+2\psi\,i)}} + 2\,\text{arc sin}\left[\frac{i}{\sqrt{2}}e^{(\frac{1}{2}\phi+\psi\,i)}\right]\right\} \qquad (3d)$$

The cusp points of this expression lie where

$$e^{(\phi+\psi\,i)} = -2;$$

that is to say where

$$\psi = \pm\ (2\,\alpha \times 1)\ \pi \text{ and } \varphi = \log 2.$$

Thus neither one lies between the limits from $\psi = +\pi$ to $\psi = -\pi$.

The function $\sigma+\tau i$ is here continuous.

Along the wall we have

$$\sigma+\tau i = \pm A\,i\left\{\sqrt{2e^{\phi} - e^{2\phi}} - 2\,\text{arc sin}\left[\frac{1}{\sqrt{2}}e^{\frac{\phi}{2}}\right]\right\}$$

If $\varphi > \log 2$, then all these values become purely imaginary, therefore $\sigma = 0$, while $\frac{d\tau}{d\varphi}$ has the value given above in equation (3c). This portion of the lines $\psi = \pm\pi$ therefore corresponds to the free portion of the jet.

If $\varphi < \log 2$ the whole expression is real up to the additive quantity $\pm A\,i\,\pi$, which latter is to be added to the value of $\tau\,i$ and $y\,i$ respectively.

The equations ($3a$) and ($3d$) correspond therefore to the outflow from an unlimited basin into a canal bounded by two planes, whose breadth is $4\,A\,\pi$ and whose walls extend from $x=-\infty$ to $x=-A\,(2-\log 2)$. The free discontinuous line of the flowing fluid curves from the nearest edge of the opening at first a little towards the side of the positive $x$, where for $\varphi=0$, $x=-A$ and reaches its greatest $x$ value when $y=\pm A\left(\frac{3}{2}\pi+1\right)$; then it turns inward towards the inside of the canal and at last asymptotically approaches the two lines $y=\pm A\,\pi$, so that finally the breadth of the outflowing jet is equal only to the half breadth of the canal.

The velocity along the discontinuous surface and at the extreme end of the outflowing jet is $\frac{1}{A}$, so that this form of motion is possible for every velocity of efflux.

I present this example especially as it shows that the form of the liquid stream in a tube can for a very long distance be determined by the form of the initial portion.

### ADDITION, BEARING ON ELECTRICAL DISTRIBUTION.

When in equation (2) we consider the quantity $\psi$ as the electric potential it gives the distribution of electricity in the neighborhood of the edges of two plane disks quite near together, assuming that their distance is indefinitely small with respect to the radius of curvature of their curved edge. This is a very simple solution of the problem that has been considered by Clausius.* It gives moreover the same distribution of electricity as he found for it; at least so far as it is independent of the curvature of the edges.

I will further add that the same method also suffices to find the distribution of electricity on two parallel, infinitely long, plane strips, whose four edges in cross section form the corners of a rectangle, that is, the cross section of the strips gives two lines which are opposite and parallel to each other. The potential function $\psi$ in this case is given by an equation of the form

$$x+y\,i=A\,(\varphi+\psi\,i)+B\frac{1}{H\,(\varphi+\psi\,i)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $H\,(u)$ represents the function designated by Jacobi in the *Fundamenta Nova*, p. 172, as the numerator of the function developed in terms of sin *am* $u$. The overlying strips correspond, according to Jacobi's notation, to the values $\varphi=\pm 2\,K$ where $x=\pm 2\,KA$ gives the half distance of the strips, while the width of the strip depends on the ratio of the constants $A$ and $B$.

The form of the equations (2) and (4) allows us to recognize that $\varphi$ and $\psi$ can be expressed as function of $x$ and $y$ only by means of most complicated serial developments.

* Poggendorff's *Annalen*, Bd. LXXXVI.

IV.

# ON A THEOREM RELATIVE TO MOVEMENTS THAT ARE GEOMETRICALLY SIMILAR IN FLUID BODIES, TOGETHER WITH AN APPLICATION TO THE PROBLEM OF STEERING BALLOONS.*

By Prof. H. VON HELMHOLTZ.

The laws of motion of cohesive and non-cohesive fluids [namely, liquids and gases] are sufficiently well known in the form of differential equations, that take into consideration not only the influence of exterior forces acting from a distance, as well as the influence of the pressure of the fluid, but also the influence of the friction [namely, both internal and external frictions, or both viscosity and resistance]. When in the application of these equations one remembers that under certain circumstances [namely, wherever a continuous motion would give a negative pressure] there must form surfaces of separation with discontinuous motion on the two sides, as I have sought to prove in a previous communication to this academy,† then will disappear the contradictions that by neglect of this consideration have hitherto been made to appear to exist between many apparent consequences of the hydro-dynamic equations on the one hand and the observed reality on the other. In fact, so far as I see, there is at present no ground for considering the hydro-dynamic equations as not being the exact expression of the laws controlling the motions of fluids.

Unfortunately it is only for relatively few and specially simple experimental cases that we are able to deduce from these differential equations the corresponding integrals appropriate to the conditions of the given special cases, especially if the nature of the problem is such that the internal friction [viscosity] and the formation of surfaces of discontinuity can not be neglected. The discontinuous surfaces are extremely variable, since they possess a sort of unstable equlibrium, and with every disturbance in the whirl they strive to unroll themselves; this circumstance makes their theoretical treatment very difficult. Thus it happens

* From the *Monatsberichte* of the Royal Academy of Berlin, June 26, 1873, pp. 501 to 514. *Wissenschaftliche Abhandlungen*, vol. II, pp. 158–171, Berlin, 1882.

† Berlin *Monatsberichte*, April 23, 1868. See also No. III of this collection of Translations.

that where we have to do practically with the motions of fluids we are thrown almost entirely back upon experimental trials, and can often, from theory, predict but very little, and that only in an uncertain manner, as to the result of new modifications of our hydraulic machines, aqueducts, or propelling apparatus.

In this state of affairs I desire to call attention to an application of the hydro-dynamic equations that allows one to transfer the results of observations made upon any fluid and with an apparatus of given dimensions and velocity over to a geometrically similar mass of another fluid and to apparatus of other magnitudes and to other velocities of motion.

To this end I designate by $u$ $v$ $w$ the components of the velocity of the first fluid in the directions of the rectangular coördinate axes $x$ $y$ $z$; by $t$ the time, by $p$ the pressure, by $\varepsilon$ the density, by $k$ its coefficient of friction (viscosity). The equations of motion in the Eulerian form introducing the frictional forces, as is done by Stokes, in case no exterior forces act upon the fluid, will now have the following form:

$$-\frac{\partial \varepsilon}{\partial t}=\frac{\partial (u.\varepsilon)}{\partial x}+\frac{\partial (v.\varepsilon)}{\partial y}+\frac{\partial (w.\varepsilon)}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$-\frac{1}{\varepsilon}\cdot\frac{\partial p}{\partial x}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-k\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right\}$$
$$-\frac{k}{3}\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

To these are still to be added the two equations that are deduced from the latter equation ($1a$) by interchanging $x$ and $u$ with $y$ and $v$ or with $z$ and $w$.

When now for another fluid the velocities are designated by $U$, $V$, $W$, the pressure by $P$, the coördinates by $X$, $Y$, $Z$, the time by $T$, the density by $E$, the viscosity constant by $K$, and if we introduce three constants $q$, $r$, and $n$, and put

$$K=qk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$E=r\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

$$U=nu \qquad X=\frac{q}{n}x$$

$$V=nv \qquad Y=\frac{q}{n}y$$

$$W=nw \qquad Z=\frac{q}{n}z$$

$$P=n^2rp+\text{constant.} \qquad T=\frac{q}{n^2}t$$

then the quantities designated by these capital letters will also fulfill the above differential equations. If we substitute these in those equations, the result, $E$, is as if all the terms of equation (1) were multiplied by

the factor $\frac{rn^2}{q}$ and all the terms of equation ($1a$) by the factor $\frac{n^3}{q}$. Of the constants $q$, $r$, $n$, two are determined through the equations (2) and ($2a$) by the nature of the fluid, but the third, $n$, is arbitrary so far as the conditions hitherto considered come into consideration.

If the fluid is incompressible, then $\varepsilon$ is to be considered as a constant and $\frac{\partial\varepsilon}{\partial t}=0$, and the above equations then suffice to determine the motion in the interior.

If the fluid is compressible, we can put

$$p=a^2\varepsilon-c \qquad (3)$$
$$P=A^2E-C \qquad (3a)$$

where $c$ and $C$ indicate constants to be added to the pressure and which have no influence on the equation $1a$.

For gases $c$ and $C$ are to be put equal to zero if the motion occurs under such circumstances that the temperature remains constant. For rapid variations of density in gases without equalization of temperature (namely non-adiabatic motions), the equations (3) and ($3a$) would only apply for the case of slight variations in density.

The equation ($3a$) is only satisfied by the above-given values for $P$ and $E$ when

$$A^2=a^2n^2.$$

By this condition therefore the third constant, $n$, is determined. The quantities $a$ and $A$ in this latter equation are the velocities of sound in the respective fluids. These quantities must change in the same ratio as the other velocities.

If the boundaries of the fluid are in part infinitely distant and in part given by moving or quiet, perfectly wetted, rigid bodies, and the coördinates and component velocities of these limiting rigid bodies are transferred from one case to the other in the same manner as has just been done for the particles of fluid, then will the boundary conditions for $U$, $V$, $W$ be fulfilled when they are fulfilled for $u$, $v$, $w$. In this I assume that on completely wetted bodies the superficial layer of fluid is held perfectly adherent; that therefore the component velocities of the surfaces of the rigid bodies and those of the adherent fluid are equal.

For imperfectly wetted solids it is as a rule assumed that there is a relative motion of the superficial fluid layers with respect to the solid. In this case the application of our principles would require that a certain ratio be assumed between the coefficients of sliding superficial friction of the fluid on the respective rigid bodies, and the internal friction (or viscosity) of the fluid.

Similarly the boundary conditions at the free surfaces of a liquid over which the surface pressure is constant, would be satisfied in case no

outside forces like gravity have an influence. But since this case occurs only in liquids [*i. e.*, fluids that form drops] that can be regarded as incompressible, therefore (for these) it is not necessary to satisfy equations (3) and (3*a*). Therefore (for these) the constant $n$ remains arbitrary, and when for this case this latter constant is so determined that $\frac{n^3}{q}=1$, then in equation (1*a*) the intensity of gravity (*i. e.*, the acceleration, $-g$) can be added to the left-hand member.

The boundary condition for a discontinuous surface is that the pressure shall be equal on both sides of such a surface, which condition will be satisfied for $P$ when it is so for $p$.

As regards the re-action of the fluid against a solid body moving in it, the pressure against the unit of area of surface increases as $n^2 r$. In the same ratio, the frictional forces increase that are proportional to the product of $k\,\varepsilon$, with the differential quotients such as $\frac{\partial u}{\partial x}$, and other similar ones. But for corresponding similar portions of the surfaces of the bounding bodies of the forces of pressure and of friction increase as

$$\frac{q^2}{n^2} \cdot n^2 . r = q^2 r.$$

The work needed to be done by the immersed bodies to overcome these resistances will therefore for equal intervals of time increase as $nq^2 r$.

In general therefore for compressible fluids [gases] and for heavy cohesive fluids [liquids under gravitation] with free surfaces, if the movement is to be completely and accurately transferred from the first fluid to the other, the three constants $n, q, r$ are completely determined by the nature of the two fluids. Only in the case of incompressible fluids without free surfaces does one constant remain indeterminate.

Now there is a large series of cases where the compressibility not only for cohesive, but also for gaseous fluids, has only an inappreciably small influence. To such cases the following considerations apply: If the constant $n$ becomes smaller while $r$ and $q$ remain unchanged, this indicates that in the second fluid the velocity of sound diminishes proportionally with $n$, and similarly for the velocities of the moving material portions, whereas the linear dimensions increase proportional to the reciprocal of $n$. For a constant value of $r$, that is to say, a constant density of the second fluid, a diminution of the velocity of sound corresponds to an increased compressibility of the fluid. Therefore with an increased compressibility, the movements remain similar. Hence it follows that when we diminish $n$, while leaving the compressibility of the fluid unchanged, the movements of the fluid themselves change and become similar to those that a more incompressible fluid would execute in a narrower space. Therefore for smaller velocities,

even in extensive spaces, the compressibility loses its influence. Under such circumstances gases move like cohesive incompressible fluids [viz, liquids], as is well known practically from many examples.

If the velocities of the material parts are in general very small, as in the case of exceedingly small oscillations, so that the course of the movement remains sensibly unchanged for a uniform increase in these velocities, then it will only be the velocity of sound that changes, and our proposition will take the following form: The sonorous vibrations of a compressible fluid can, in larger spaces, behave mechanically the same as more rapid oscillations of a less compressible fluid in smaller spaces. An example of the utilization of the similarity here spoken of is found in my investigations on the acoustic movement at the ends of open organ pipes.* In that study the possibility of replacing the analytical conditions of the motion of the air by the simpler ones of the motion of water depended on the principle that the dimensions of the given spaces must be very small in comparison to the wave lengths of the existing acoustic vibrations.

On the other hand the viscosity also shows itself less influential in the movements of fluids in large spaces. If we let $n$ remain unchanged while $q$ increases we obtain the same ratio between the frictional forces and the pressure forces. That is to say, if we increase the dimensions and the friction constants in the same ratio, then the movements in the enlarged system remain similar so long as the velocities do not change. Hence it follows that in such an enlarged model, when the friction constant is not increased in the same ratio, but remains unchanged, the friction loses in influence for the same velocity. That which holds good for greater dimensions with unchanged velocities also obtains for increased velocities with unchanged dimensions. For one can also simultaneously let $n$ increase proportional to $q$.

In fact, in most practical experiments in extended fluid masses, the resistance that arises from the accelerations of the fluid,† and especially in consequence of the formation of surfaces of discontinuity is by far the most important. Its magnitude increases proportionally to the square of the velocity, whereas the resistance depending upon the friction proper (internal friction or viscosity and surface-hesion), which increases simply in proportion to the velocity, becomes appreciable only in experiments in very narrow tubes and vessels.

Neglecting the friction, that is to say, if in the above equations we put the constants

$$k = K = 0$$

then will the constant $q$ also become arbitrary, and we can change the dimensions and velocities in any ratio whatever.

If however the force of gravity comes into consideration as in the

* Borchardt's *Journal für Mathematik*, 1859, vol. LVII, pp. 1-72.

† [These resistances are those that I have called "convective" in my Treatise on Meteorological Methods and Apparatus.—*C. A.*]

case of waves on the free surface of water, then, according to the remarks already made the ratio $\frac{n^3}{q}$ must remain unchanged, therefore $q$ must be put $=n^3$. Then will

$$\begin{matrix} X=n^2x \\ Y=n^2y \\ Z=n^2z \end{matrix} \quad T=nt.$$

Therefore when the wave lengths increase in the ratio $n^2$ the duration of the oscillations will increase only in the ratio $n$, which corresponds to the well-known law of the velocity of propagation for the surface waves of water, which velocity increases as the square root of the wave length. Thus this result is attained very simply and for all wave forms, without the necessity of knowing a single integral of wave motion.

The same principle is applicable to the relative resistances that ships having $n^2$ times the dimensions and $n$ times the velocity, experience by reason of the waves that they excite on the surface of the water. The total resistance in this case increases as $q^2r$, and since for the same fluid $r=1$ therefore the resistance increases as $n^6$ and the work needed to overcome it as $n^7$, therefore in a rather larger ratio than the volume of the ship, while the supply of fuel and the size of the boiler that must do the work can increase only in the same ratio as the volume of the ship, namely as $n^6$. Therefore so long as lighter machinery can not be applied (including the supply of coal) the velocity of such an enlarged ship can increase above a certain limit only by a ratio that is smaller than that of the square root of the increase of the linear dimensions.

A similar computation holds good for the model of the bird in the air. When we increase the linear dimensions of a bird and would take into consideration the viscosity, we must put $q$ and $r$ equal to unity because the medium, namely the air, remains unchanged. Let $n$ be a vulgar fraction, then will the velocity be reduced in the same proportion as the volume of the bird increases and the pressure (of the air) against the total surface of the larger bird will only attain the same value as for the smaller bird, therefore will not be able to bear up the weight of the larger bird.

If we allow ourselves to neglect the friction, which according to the above remarks we can do so much the more readily the more we increase the dimensions, or for the same dimensions increase the velocities, then $q$ is arbitrary and the change of dimensions and velocities must be so made that the total pressure against the surfaces shall increase as the weight of the body or we must have $q^2=\frac{q^3}{n^3}$ or $q=n^3$. In order to execute the corresponding motions, the work that will be necessary will be

$$q^2n=n^7=\left(\frac{q}{n}\right)^{\frac{7}{2}};$$

but the volume of the body and of the muscles that do the work increases only in the ratio $\left(\frac{q}{n}\right)^3$

Hence it follows that the size of a bird has a limit, unless the muscles can be further developed in such a manner that for the same mass as now they shall perform more work. Now it is precisely among the larger birds, that are capable of the greater performances in flying, that we find those that eat only flesh and fish; they are animals that consume concentrated food and need no extensive system of digestive organs. Among the smaller birds many grain eaters like doves and the smaller singing birds are also good flyers. It therefore appears probable that in the model of the great vulture, nature has already reached the limit that can be attained with the muscles as working organs, and under the most favorable conditions of subsistence, for the magnitude of a creature that shall raise itself by its wings* and remain a long time in the air.

Under these circumstances it is scarcely to be considered as probable that man even by means of the most ingenious wing-like mechanism that must be moved by his own muscles will ever possess the strength needed to raise his own weight in the air and continue there.

As concerns the question as to the possibility of driving balloons forward relative to the surrounding air, our propositions allow us to compare this problem with the other one that is practically executed in many forms, namely, to drive a ship forwards in water by means of oar-like or screw-like organs of motion. In studying this we must not consider movement on the surface, but rather imagine to ourselves a ship driven along under the surface. But such a balloon which presents a surface above and below that is congruent with the submerged surface of an ordinary ship scarcely differs in its powers of motion from an ordinary ship.

If now we let the small letters of the two above given systems of hydro-dynamic equations refer to water and the large letters to the air, then for 0° temperature and 760 mm. of the barometer, we have

$$\frac{1}{r}=773$$

According to the determination of O. E. Meyer and Clerke Maxwell,

$$q=0.8082$$

the velocity of sound gives for $n$ the value

$$n=0.2314$$

Hence the increase of linear dimensions is

$$\frac{q}{n}=3.4928$$

---

*[That is, by the work done by its wings; this of course does not cover the case of soaring birds whose muscles do no lifting work but simply keep the wings in the best position for the wind to act on them.—*C. A.*]

and the increase of volume is

$$\left(\frac{q}{n}\right)^3=42.61$$

The work in this case is very slight, namely,

$$q^2nr=\frac{1}{5114.3}$$

The ship, including the crew and the load, must weigh as much as the volume of water displaced by it. The balloon, filled with hydrogen, in order to carry an equal weight with the ship, must have a volume 837 times as great. If it is filled with illuminating gas of a specific gravity 0.65 relative to that of the air, it must have a volume 2,208.5 times as great as the ship. Thus, the weight that the balloon must have for the given dimension is now determined. The weight for the hydrogen balloon would be $\frac{42.6}{837}=\frac{1}{19.6}$ that of the ship; that of the illuminating gas balloon would be $\frac{42.6}{2208.5}=\frac{1}{51.8}$ that of the ship.

The work that is necessary under such circumstances to propel the balloon, as the above number for the value of $q^2nr$ shows, would, however, for the adopted small velocity, be reduced in much greater proportion than that of the weight of the balloon to the weight of the ship, so that the work here required for the given weight is easy to accomplish in the balloon. For even when we so choose the ship that its load in excess of that of the driving machine (or in excess of the men who act as the machine) is negligible, then the weight of the illuminating-gas balloon need be only $\frac{1}{52}$ part of the weight of this driving machine, but the machine thus carried by it would also have to do only the one $\frac{1}{5114}$ of the work of the ship's machine, it would, therefore, need to have a less weight in about this latter ratio. Especially would this latter be the case when we utilize men as the driving machine, whose work and weight both increase proportionally to the number.

So far we can therefore apply the transference from ship to balloon with complete consideration of the peculiarities of air and water. As a maximum velocity for fast ships (large naval steamers), "The Engineer's Pocket Book," published by the society "*Die Hütte*," gives 18 feet per second, or 2.7 German miles, or 21 kilometers per hour. Similarly built balloons, with relatively very feeble or small propelling machinery, can attain about one-fourth of this velocity.

Ships of the above-given dimensions find the limit of their efficiency bounded by the limits of the power of the machinery (including the fuel) that they can carry. However, the practical experience thus far attained allows us to neglect the influence of viscosity for large, swift

ships, and therefore to arbitrarily assume the constant $q$, as also $n$ (when we can neglect the movements at the surface). If we assume that $q$ increases proportionally to $n$, then the dimensions remain unchanged, the velocities increase as $n$, the resistance as $n^2$, the work done as $n^3$. If therefore we were able to build a marine engine of the same weight as the present ones, but of greater efficiency, we would then be able also to attain greater velocities.

We must compare the balloon with such a ship, although the latter has not yet been constructed, in order to attain complete utilization of the propelling machine that goes up with it. But for this case also and for unchanged dimensions, when the velocity increases as $n$ the work must increase as $n^3$.

Now the ratio between weight and work done by the men who are carried by a balloon can only, for balloons of very large dimensions, be perhaps more favorable than for a war ship and its machinery. For the latter I compute from the technical data that to attain a velocity of 18 feet requires an expenditure of one horse-power to 4636.1 kilograms weight.* On the other hand, a man weighing 200 pounds, who under favorable circumstances can do 75 foot-pounds of work per second during eight hours daily, gives on the average for the day one horse-power per 1,920 kilograms. When therefore the balloon weighs one and a half times as much as the laboring men whom it carries, then the ratio is the same as for the ship. Dupuy de Lôme has carried out his experiments under somewhat less favorable circumstances; in his balloon were a crew of 14 men whose weight was one-fourth of the whole, and of whom only eight worked. Under these circumstances it is a relatively very favorable assumption when for the balloon we assume the ratio between the weight and the work to be the same as for a war steamer. We can therefore for the illuminating gas balloon increase the ratio $\frac{51.831}{5114n^3}$ between work and weight by increasing $n$ so that the ratio shall equal unity; that is to say, equal to the value for ships. In this case we must have

$$n=4.6208.$$

Since now the velocity $U$ of the balloon which we have before computed under the assumption of a perfect geometrical similarity in the

*The special data on which the computation is based are as follows:

$L$ = length of the ship over all = 230 Prussian feet.

$B$ = breadth of the ship over all = 54 " "

$H$ = total height of the ship = 24 feet.

$T$ = depth under water = $H - \frac{1}{8} B$

$V$ = volume of water displacement = 0.46 $L. B. T.$

Weight of one cubic foot of sea water = 63.343 lbs.

A the area of the immersed principal section = 1000 sq. feet.

The total work = $\zeta A V^3$

Where $\zeta = 0.46$.

movements has only 0.2314 that of the velocity $u$ of the ship, therefore there results:

$$U=0.2314.n.u=1.06925u.$$

For the hydrogen balloon under the same assumptions the velocity will be somewhat larger, since in this case we have to assume

$$\frac{19.6}{5114}n^3=1.$$

Hence,

$$n=6.390$$
$$U=0.2314.n.u=1.4786u.$$

which is nearly one and a half times the velocity hitherto attained in naval steamers. This last velocity for a hydrogen balloon would suffice to go slowly forwards against a fresh breeze.

But it is to be remarked that these computations relate to colossal balloons whose linear dimensions are three and a half times larger than those of the immersed portion of a large man-of-war, and that the inflammable gas balloon would weigh 60220 kilograms, while that of Dupuy de Lôme only weighed 3799 kilograms. In order to return to dimensions that are attainable in actual practice, one must so diminish $q$ and $n$ as that the ratio of the work to the weight shall remain unchanged, therefore, so that

$$q^2 n : \left(\frac{q}{n}\right)^3 = 1,$$

whence

$$q = n^4.$$

In this way the velocity $n$ will diminish as the cube root of the linear dimensions or as the ninth root of the volume or the weight. This reduction is relatively unimportant. If we pass, for example, from our ideal balloon down to one of the weight of that of Dupuy, there results a reduction of the velocity in the ratio of 1.36 to 1; this would give a velocity of 14.15 feet per second, or 16.5 kilometres per hour. The linear dimensions of the balloon would therefore exceed in the ratio 1.4 to 1 the dimensions of the ship that is compared with it.

The ratio between work and load in Dupuy's experiments correspond to the above assumptions very nearly. The eight men that worked for him are, according to our previous estimate, to be put down at 800 kilograms, which is rather more than one-fifth of the total weight. Since however the experiment only lasted a short time, therefore these men could work the whole time through with their whole energy, whereas in our computation only the average value of eight hours of work is assumed for the whole day. Therefore these eight men are equal to twenty-four steady workers, whereby the difference is more than made up. Dupuy gives, as having been attained independent of the wind,

on the average 8 kilometers per hour for the whole duration of the experiment, and $10\frac{1}{4}$ kilometers attained by intense work. He is therefore not very far behind the limit that my computations show attainable with a balloon of such dimensions.

In the preceding computation we have however only taken account of the ratio between the effective force and the weight, and have assumed that the form of such a balloon and of its motor can be attained with the materials at our disposal. But here seems to me to lie one of the principal difficulties of the practical execution. For the parts of a machine made of rigid bodies do not by a geometrically similar increase in their linear dimensions retain the necessary stiffness; they must be made thicker, and therefore heavier. If on the other hand with small motors one would attain the same effect, by means of greater velocity, then work is dissipated. The pressure against the whole surface of a motor (a ship's propeller, or oars or paddles) increases as $q^2 r$. If this pressure, which determines the propelling force, is to remain unchanged, we can only diminish the dimensions in so far as we increase $n$, and therefore also the velocities; but then the work increases also as $q^2 n r$, and therefore proportionally to $n$. Therefore one can work economically only with relatively slow-moving motors of large surface. And to realize this in the necessary dimensions without too great a load for the balloon will be one of the greatest practical difficulties.

V.

## ON ATMOSPHERIC MOVEMENTS.*

(*FIRST PAPER.*)

By Prof. H. von Helmholtz.

### I. INFLUENCE OF VISCOSITY ON THE GENERAL CIRCULATION OF THE ATMOSPHERE.

The influence of fluid friction in the interior of very extended regions that are filled with fluid and contain no vortex motion is always relatively very small. This can be proved from considerations that are based upon the principle of mechanical similarity. If we form the Eulerian hydro-dynamic equations and in them indicate by $u$, $v$, $w$ the components of the velocity parallel to the axes of $x$, $y$, $z$; by $\varepsilon$ the density, by $p$ the pressure, by $P$ the potential of the forces that act upon a unit of mass of the fluid; then if we consider $P$, $\varepsilon$, $p$, $u$, $v$, $w$ as functions of $x$, $y$, $z$, $t$ we have, as is well known, the following partial differential equations for a fluid under the influence of friction†:

$$-\frac{\partial P}{\partial x}-\frac{1}{\varepsilon}\frac{\partial p}{\partial x}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\frac{k^2}{\varepsilon}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \quad . \quad . \quad . \quad . \quad (1)$$

$$-\frac{\partial \varepsilon}{\partial t}=\frac{\partial(\varepsilon u)}{\partial x}+\frac{\partial(\varepsilon v)}{\partial y}+\frac{\partial(\varepsilon w)}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

Two other equations symmetrical with regard to the other coördinates are to be added to the first of these equations. If now we have found any special integral whatever of these equations, which obtains for a definite region, then the equations will also hold good for a second case where all the linear dimensions $x$, $y$, $z$ and also the time $t$ and the friction constant $k^2$ are increased by a factor $n$, but where $P$, $p$, $\varepsilon$, $u$, $v$, $w$ retain for every value of the new coördinates $nx$, $ny$, $nz$, $nt$, the same values as they had in the first case for the original coördinates $x$, $y$, $z$, $t$. Hence it follows that when in the movement of the magnified mass the friction constant can be also simultaneously and correspondingly increased, the

---

* From the *Sitzungsberichte* of the Royal Prussian Academy of Science at Berlin, 1888, May 31, pp. 647–663.

[† Namely viscosity as represented by Maxwell's kinematic coefficient $\nu$ or Helmholtz' $\frac{k^2}{\varepsilon}=\frac{0.0001878}{0.001293}=0.13417$]

movement takes place in an analogous manner, only slower. When this is not the case and when the friction retains its value unchanged then will the influence of the friction on the increased mass be very much less than upon the smaller mass. In consequence of this the greater mass will show the effects of its inertia as influenced much less by friction.

It is to be remarked that the potential $P$ remains unchanged by the increase of the mass, but the force $\frac{\partial P}{\partial x}$ is reduced to $\frac{1}{n}$ of its value and that the whole process as already remarked requires for its completion $n$ times the time.

Since the density and pressure are to remain unchanged therefore also any temperature differences that are present retain their magnitude and influence and do not disturb the relations implied in the mechanical similarity.

Unfortunately we can not imitate in small models the varying density of the atmosphere at different altitudes since we can not correspondingly change the force of gravity that is included in the expression $\frac{\partial P}{\partial x}$. Our mechanical comparisons are only able to imitate an atmosphere of constant density. Such an one must, as is well known, have an altitude of 8026 metres at 0° C. in order to produce the mean barometrical reading of 76 centimetres of mercury. If we desire in a model to represent the atmosphere by a layer of one metre in altitude, then we would need to reduce the day to 10. 8 seconds, or the year to 65. 5 minutes, and the influence of friction in movements at velocities that correspond to those of the atmosphere would in a small model be 8026 times as great as in the atmosphere. The loss of living force in the atmosphere during a year would therefore correspond to that lost in our model in $\frac{65.5}{8026}$ of a minute, which corresponds to less than a half a second.

On the other hand it is possible with the measured value of the friction constant of the air to compute for some simple cases how long a time would be required in order to reduce to one-half of its velocity any motion that is hindered only by internal friction. In this case the assumption of a constant density is for our purpose more unfavorable than the adoption of the actual variable density.

Assume that a stratum of air whose constant density is such as that of the lower stratum of the atmosphere, spreads over an unlimited plane and has a forward movement whose velocity is $u$ in the direction of $x$ parallel to the plane. Let $z$ be the vertical coördinate, then the equation of motion for the interior of the mass is

$$\frac{\partial u}{\partial t} - \frac{k^2}{\varepsilon} \cdot \frac{\partial^2 u}{\partial z^2} = 0 \quad \ldots \ldots \ldots \quad (2)$$

Assume that the fluid adheres to the earth's surface where $z = 0$, therefore for this surface we have

$$u = 0_{z=0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

At the upper boundary surface where $z = h$ the fluid experiences no friction, therefore for that surface we have

$$\frac{\partial u}{\partial z} = 0_{z=h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

Of the special integrals of the equation (2) that fulfill the boundary condition (2$a$), namely:

$$u = Ae^{-nt} \sin (qx)$$
$$n = \frac{k^2}{\varepsilon} q^2$$

the one that also fulfills the condition ($2_b$) and is the most slowly diminishing is given by the value

$$q = \frac{\pi}{2h}$$

Hence follows

$$n = \frac{k^2}{\varepsilon} \cdot \frac{\pi^2}{4h^2}$$

The factor $e^{-nt}$ becomes 1 at the time $t=0$: in order that this factor may be equal to one-half we must have

$$nt = \text{nat. log. } 2 = 0.69315.$$

According to Maxwell's determinations (Theory of Heat, London 1871, p. 279, where $\frac{k^2}{\varepsilon}$ is expressed by $\nu$ and $k^2$ by $\mu$), we have

$$\frac{k^2}{\varepsilon} = 0.13417 \; [1 + 0.00366\theta_c] \cdot \frac{[\text{centimetre}]^2}{\text{second}}$$

where $\theta_c$ indicates the temperature centigrade. From this there results, for the temperature 0° C.,

$$t = 42747 \text{ years.}$$

If we distribute the same mass of air throughout a thicker stratum with less density so that $\varepsilon \, . \, h$, as also the $k^2$ which is independent of $\varepsilon$, retains its value unchanged, then $t$ must increase with $h$. Hence it follows that in the upper thinner strata of the atmosphere the effect of viscosity propagates itself through atmospheric strata of equal mass more slowly than through the lower denser strata.

On the other hand an increase of the absolute temperature $\theta$ will cause the time $t$ to diminish as $\frac{1}{\theta}$. The lower temperature of the upper

strata of the atmosphere also diminishes the effect of the viscosity here under consideration.

This computation also shows how extremely unimportant for the upper strata of the air are those effects of viscosity that can arise on the earth's surface in the course of a year.

Only at the fixed boundaries of the space that the atmosphere fills, or at the interior surfaces of discontinuity where currents of different velocity border on each other, do the surface forces remain the same when the scale of dimensions is increased and the coefficient of friction is not simultaneously increased, and this allows us to recognize that the annulment of living force by viscosity can take place principally only at the surface of the ground and at the discontinuous surfaces that occur in vortex motions.

A similar relation obtains with regard to those temperature changes that can be effected by the true conduction of heat in the narrower sense, namely, the diffusion of moving molecules of gas between the warmer and colder strata. The coefficient $\varkappa$ of conduction for heat, when we choose as the unit of heat that which warms a unit volume of the substance by one degree in temperature (or the thermometric coefficient of conduction), is, according to Maxwell (Theory of Heat, page 302):

$$\varkappa = \frac{5}{3.\gamma} \cdot \left(\frac{k^2}{\varepsilon}\right)$$

where $\gamma$ is the ratio between the two specific heats of gases.

In order to solve the corresponding problem for the conduction of heat this $\varkappa$ is to be substituted in equation (2) instead of $\frac{k^2}{\varepsilon}$, and if we put $\gamma = 1.41$ it is seen that in the above-assumed atmosphere of uniform density under a pressure of 76 centimetres of mercury and at a temperature of 0° an interval of 36164 years would be necessary in order by conduction to reduce by one-half the final difference in temperature of the upper and lower surfaces. Therefore also in the interchange of heat only its radiation and its convection by the motion of the air need be taken into consideration, except at the boundary between it and the earth's surface and at the interior surfaces of discontinuity.

On the other hand, simple computations have frequently shown that an unrestricted circulation of the air in the trade zones can not exist even up to 30° latitude.

If we imagine a rotating ring of air whose axis coincides with that of the earth and which, by the pressure of neighboring similar rings, is pushed now northward and now southward, and in which we can neglect the friction, then, according to the well-known general mechanical principle, the moment of rotation of this ring must remain constant. We will indicate this moment as computed for the unit of mass by $\Omega$,

and the angular velocity of the ring by $\omega$, and its radius by $\rho$; then, as is well known,

$$\Omega = \omega \rho^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and therefor $\omega$ must vary inversely proportionally with $\rho^2$. If we indicate the mean radius of the earth by $R = 6379600$ metres, the geographical latitude by $\beta$, and the velocity of diurnal rotation of the earth by $\omega_0$, then the corresponding relative velocity at the earth's surface for a ring of air that preserves a calm at the equator is

$$\rho (\omega - \omega_0) = \omega_0 \left[ \frac{R}{\cos \beta} - R \cos \beta \right].$$

For air that is resting quietly at the equator in the zone of calms and is thence pushed up to the latitude of 10°, this expression gives the acquired wind velocity 14.18 metres per second, and similarly for air pushed up to latitude 20°, 57.63 metres, and for 30°, 133.65 metres per second.

Since 20 metres per second is the velocity of a railroad express train, therefore these numbers show without further consideration that such gales do not exist over any broad zone of the earth. We therefore ought not to make the assumption that the air which has risen at the equator reaches the earth's surface again unchecked in its motion even 20° farther northwards.

The matter is not much better if we assume the atmospheric ring resting at some intermediate latitude. In that case it would give an east wind at the equator, but a west wind at 30° latitude; but both velocities would far exceed the ordinary velocities of the observed winds.

Since now in fact observations do demonstrate a circulation of the air in the trade-wind zone, therefore the question recurs: By what means is the west-east velocity of this mass of air checked and altered? The resolution of this question is the object of the following remarks:

## II. ON THE EQUILIBRIUM OF ROTATING RINGS OF AIR AT DIFFERENT TEMPERATURES.

If we introduce into equations (1) only rotatory motions about the axis, whereby $\omega$, $\Omega$, and $\rho$ retain the significance just given them we then have

$$u = 0$$

$$v = -z\,\omega = -z.\frac{\Omega}{\rho^2}$$

$$w = y\,\omega = y.\frac{\Omega}{\rho^2}$$

and if we consider a steady mode of motion, in which $\Omega$, $p$, $P$, and $\varepsilon$ are functions of $x$ and $\rho$ only, then the equations (1) become

$$-\frac{\partial P}{\partial x}-\frac{1}{\varepsilon}.\frac{\partial p}{\partial x}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

$$-\frac{\partial P}{\partial \rho}.\frac{y}{\rho}-\frac{1}{\varepsilon}.\frac{\partial p}{\partial \rho}.\frac{y}{\rho}=-y.\frac{\Omega^2}{\rho^4}$$

$$-\frac{\partial P}{\partial \rho}.\frac{z}{\rho}-\frac{1}{\varepsilon}.\frac{\partial p}{\partial \rho}.\frac{z}{\rho}=-z.\frac{\Omega^2}{\rho^4}.$$

The two last equations combine into the one following:

$$\frac{\partial P}{\partial \rho}+\frac{1}{\varepsilon}.\frac{\partial p}{\partial \rho}=\frac{\Omega^2}{\rho^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

Equation $1_a$ is satisfied by the above adopted values of $u, v, w$. Therefore the only equations to be satisfied are ($3a$) and ($3b$).

As concerns the value of the density $\varepsilon$, this depends upon the pressure $p$ and the temperature $\theta$. Since appreciable effective conduction of heat is excluded, therefore we must here retain the law of adiabatic variations between $p$ and $\varepsilon$; therefore we have

$$\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}=\frac{\varepsilon}{\varepsilon_0},$$

wherein $\gamma$ again represents the ratio of the specific heats. If we indicate by $\theta$ the temperature that the mass of air under consideration would acquire adiabatically under the pressure $p_0$ (wherefore $\theta$ indicates the constant quantity of heat contained in the air while its temperature is varying with the pressure), and if we put

$$\frac{p_0}{\varepsilon_0\theta}=\Re$$

then we have

$$\frac{1}{\varepsilon}\cdot\frac{\partial p}{\partial \rho}=\left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}}\cdot\frac{\theta.\Re}{p_0}\cdot\frac{\partial p}{\partial \rho};$$

or if, for further abbreviation, we put

$$\frac{\gamma}{\gamma-1}.\Re\cdot p^{\frac{1-\gamma}{\gamma}}=q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

$$p^{\frac{\gamma-1}{\gamma}}=\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3d)$$

we shall have

$$\frac{1}{\varepsilon}.\frac{\partial p}{\partial \rho}=q\cdot\theta\cdot\frac{\partial \pi}{\partial \rho},$$

wherein $q$ indicates a constant peculiar to the gas and independent of $\theta$ and $p$. Similarly we also have

$$\frac{1}{\varepsilon} \cdot \frac{\partial p}{\partial x} = q\theta \frac{\partial \pi}{\partial x}$$

and therefore within a stratum of air having a constant $\theta$ and $\Omega$ we have, according to equations (3$a$) and (3$b$),

$$P + q \cdot \theta \cdot \pi = -\frac{1}{2} \cdot \frac{\Omega^2}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3e)$$

The very slight deviation of the earth from a spherical form allows us to simplify the computation on the one hand by regarding the earth's surface as a sphere, but on the other hand by giving the potential $P$ an addition, the effect of which is that for the normal velocity of rotation $\omega_0$ of the earth, its spherical surface becomes a level surface. To this end we put

$$P = -\frac{G}{r} + \frac{1}{2}\omega_0^2 \rho^2,$$

[Where $G$=normal force of gravity: $r$=distance from center of gravity to point or stratum in the actual atmosphere.]

This gives the component in the direction of $x$, of the forces acting upon the unit of mass,

$$X = -\frac{\partial P}{\partial x} = -\frac{Gx}{r^2};$$

and, for the component in the direction of $\rho$,

$$P = -\frac{\partial P}{\partial \rho} = -\frac{G.\rho}{r^3} - \omega^2 \rho$$

If to the latter the centrifugal force $+\omega^2.\rho$ is also added, there remains only one force on the rotating earth and which is directed normal to the spherical surface. Thus the spherical surface becomes the level surface of the combined potential force and centrifugal force, as indeed the surface of the earth really is.

Thus our equation (3$e$) becomes

$$q.\theta.\pi = -\frac{1}{2} \cdot \frac{\Omega^2}{\rho^2} + \frac{G}{r} - \frac{1}{2}\omega_0^2 \rho^2 + C \quad . \quad . \quad . \quad . \quad (3f)$$

The function $\pi$ which is some power of the pressure $p$ with positive exponent, increases and diminishes with $p$, and remains unchanged when $p$ remains unchanged, so that we can determine the direction of the changes of the pressure easily by the changes of $\pi$.

Within a uniform stratum and with unchanged $r$, that is to say, for

a constant elevation above the earth's surface, $\pi$ has a maximum value at the station and latitude where

$$\frac{\Omega^2}{\rho^3}=\omega_0^2\rho;$$

or, if we introduce $\omega$ instead of $\Omega$ from equation (3), the maximum occurs where

$$\omega^2=\omega_0^2;$$

that is to say, where the [movement of the] ring causes a calm [on the earth's surface]. Towards this locality the pressure increases both from the pole and from the equator.

### III. EQULIBRIUM BETWEEN ADJACENT STRATA HAVING DIFFERENT VALUES OF $\theta$ AND $\Omega$.

On both sides of the surfaces separating such strata, $p$ and therefore also $q\ \pi$ (see equation $3d$) must have the same value. If we distinguish the quantities on either side [of the boundary surface] by the indices 1 and 2 we obtain from equation ($3f$)

$$\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)\cdot\frac{G}{r}=\tfrac{1}{2}\cdot\frac{1}{\rho^2}\left[\frac{\Omega_1^2}{\theta_1}-\frac{\Omega_2^2}{\theta_2}\right]+\tfrac{1}{2}\omega_0^2\rho^2\left[\frac{1}{\theta_1}-\frac{1}{\theta_2}\right]-\frac{C_1}{\theta_1}+\frac{C_2}{\theta_2}\quad . \quad . \quad (4).$$

This should be the equation of the boundary curve, linear with respect to $r$ and quadratic with respect to $\rho^2$.

In order to find the direction of the tangent to this curve we differentiate equation (4) with respect to $r$ and $\rho$, whence we get

$$\frac{G}{r^2}dr=\frac{d\rho}{\rho^3}\left[\frac{\Omega_1^2\theta_2-\Omega_2^2\theta_1}{\theta_2-\theta_1}-\omega_0^2\rho^4\right]\quad . \quad . \quad . \quad . \quad . \quad (4a)$$

or, if instead of $\Omega$ we introduce the corresponding value of $\omega$ from equation (3),

$$+\frac{G}{r^2}dr=\rho.d\rho.\ \frac{(\omega_2^2-\omega_0^2)\theta_1-(\omega_1^2-\omega_0^2)\theta_2}{\theta_1-\theta_2}\quad . \quad . \quad . \quad . \quad (4b).$$

In order to decide how the two layers must lie with respect to the boundary surface if they are to have stable equilibrium, we reason as follows: The equation of the boundary surface (4) can, in accordance with the method of its deduction, be also written

$$\pi_1-\pi_2=\text{constant}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4c);$$

or, if we designate by $ds$ one of its elements of length,

$$\frac{\partial}{\partial s}[\pi_1-\pi_2]=0.$$

Now $\pi_1$ and $\pi_2$ are functions that also have a meaning when continued beyond the boundary curve, and can be so extended by continuous

change [*i. e.*, without discontinuity]. The difference $(\pi_1 - \pi_2)$ will therefore in general increase on one side of the surface for increasing distance $dn$ from this surface, but decrease, that is to say, become negative, on the other side; and thus on the side where $\frac{d(\omega_1-\omega_2)}{dn}$ is positive we must have $\frac{\partial}{\partial h}(\pi_1-\pi_2)>0$ or positive for every other direction $dh$, in which one moves from any point of the surface towards the same side as $dn$.

If $dh$ is drawn toward the other side of the surface for which $\pi_1-\pi_2=0$, then will

$$\frac{\partial}{\partial h}(\pi_1-\pi_2)<0, \text{ or negative.}$$

If now the difference is positive on that side of the surface designated by the subscript index 1, then in case there is an infinitely small protrusion of the boundary surface toward this side, this protrusion will be pressed back by the exterior and greater $\pi_1$; similarly an infinitely small protrusion toward the negative side will also be pushed back, since there, on the other hand, $\pi_1$ diminishes more rapidly in the interior of such protrusion. Therefore in both these cases the equilibrium is stable. On the other hand, the equilibrium is unstable when the difference $(\pi_1-\pi_2)$ on the side of $\pi_1$ is negative.

Now we need not form the differential quotients for the direction $dn$. It suffices to form them for $dr$ or $d\rho$, and to merely determine whether the positive $dr$ or $d\rho$ look toward the side whose index is 1 or that whose index is 2.

By forming these differential quotients from the equation ($3f$) there results

$$q\cdot\frac{\partial(\pi_1-\pi_2)}{\partial r} = -\frac{G}{r_2}\left[\frac{1}{\theta_1}-\frac{1}{\theta_2}\right] \quad . \quad . \quad . \quad . \quad (4d).$$

The differential quotient is positive when $\theta_1 > \theta_2$. The partial differentiation with respect to $r$ while $\rho$ remains unchanged, indicates a progress in an ascending direction parallel to the earth's axis; that is to say, in the direction of a line pointing towards the celestial pole.

*The equilibrium is stable when the strata containing the greater quantity of heat lie at higher elevations on the side towards the celestial poles.*

We now form the other differential quotients

$$q\cdot\frac{\partial}{\partial\rho}(\pi_1-\pi_2)=\frac{1}{\rho^3}\left(\frac{\Omega_1^2}{\theta_1}-\frac{\Omega_2^2}{\theta_2}\right)-\omega_0^2\rho\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right) \quad . \quad . \quad . \quad . \quad (4e).$$

$$=\rho\left[\frac{\omega_1^2-\omega_0^2}{\theta_1}-\frac{\omega_2^2-\omega_0^2}{\theta_2}\right] \quad . \quad . \quad . \quad . \quad . \quad (4f).$$

If in these equations $\theta_1$ indicates the greater quantity of heat, then the equilibrium is stable when everywhere along the boundary surface we have

$$\rho\frac{\omega_1^2-\omega_0^2}{\theta_1} > \rho\cdot\frac{\omega_2^2-\omega_0^2}{\theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4g).$$

Both these values are positive where the west wind prevails; both negative where the east wind prevails.

The equation (4c) can also be written

$$q \cdot \frac{\partial}{\partial \rho}(\tau_1 - \tau_2) = \frac{1}{\rho^3} \cdot \frac{\tau_1 - \tau_2}{\tau_1 \tau_2}\left[\omega^2 \rho^4 + \frac{\Omega_1^2 \tau_2 - \Omega_2^2 \tau_1}{\tau_1 - \tau_2}\right].$$

In order that this may be positive at all latitudes, the following inequality must be satisfied

$$\Omega_1^2 \tau_2 > \Omega_2^2 \tau_1$$

or,

$$\frac{\Omega_1^2}{\tau_1} > \frac{\Omega_2^2}{\tau_2}.$$

Ordinarily this will be the case, since in general $\tau$ increases simultaneously with $\rho$ and from a definite value at the pole to a finite value at the equator. Similarly $\Omega_1^2$ also increases with $\rho$, and from zero at the pole to $\omega_0^2 \rho^2$ at the equator, so that $\frac{\Omega^2}{\tau}$ also increases from zero at the pole to a definite positive value at the equator. We will therefore designate this case as the normal case. Exceptions can only occur under special conditions within limited zones.

In the normal case as we progress along the same level, the warmer $\tau_1$ lies on the side of the greater $\rho$: that is to say, on the side towards the equator, and equally on the side of the greater $r$ if we progress toward the celestial pole; that is to say, $\rho$ and $r$ increase toward the same side of the boundary surface, and this surface must be so inclined that the tangent of its meridian section intersects the celestial sphere between the pole and the point of the horizon lying immediately beneath it. Near the equator, where the pole rises very little above the horizon, this gives an inclination to the boundary surface such that it makes a very small acute angle with the horizon.

In accordance with this, equation (4a) shows us that under those circumstances $\frac{dr}{d\rho}$ is negative along the boundary surface itself.

Therefore the normal inclination of the bounding surface is in an ascending direction toward a point situated beneath the celestial pole.

If on the other hand exceptional localities should exist at which

$$\omega^2 \rho^4 + \frac{\Omega_1^2 \tau_2 - \Omega_2^2 \tau_1}{\tau_1 - \tau_2} < 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

then in such cases according to equation (4a) $\frac{dr}{d\rho}$ will be positive; that is to say, the boundary line will ascend to higher levels as we depart from the earth's axis.

Since moreover equation (4d) shows that as we proceed in the direction of a line drawn to the pole, the warmer air must lie higher, there-

fore this line can not twice intersect the boundary surface between two layers, and consequently in the abnormal case this line must necessarily lie between the boundary surface and the horizontal plane located at the pole. Therefore the tangents to the meridional section of the boundary surfaces must intersect the greater arcs on the celestial sphere somewhere between the pole and the equatorial side of the horizon.

The smaller the difference of temperature is relative to the difference of the velocities of rotation so much the nearer does the tangent just referred to approach the pole.

Moreover at different points of the bounding line of the same two layers there can occur both normal and abnormal inclinations. For since in the expression (see equation 4*h*) on whose positive or negative value such occurrence depends, the $\Omega$ and $\theta$ throughout the extent of each layer are constant, therefore for the same altitude above the earth this value can have a positive value near the equator but a negative value near the poles. Between these the boundary curve must attain a maximum altitude where the quantity under consideration passes from positive through zero to negative. At this place also, according to equation (4*a*), we have $\frac{dr}{d\rho} = 0$, therefore $r$ is a limiting value and is here a maximum.

*Location of the strata in the case when the velocity of rotation varies continuously with the quantity of heat contained.*—The considerations hitherto set forth can also be extended to the case where $\Omega$ is a continuous function of $\theta$, and the value of $\theta$ in the atmospheric strata is continually changing. The individual strata are in this case to be considered as indefinitely thin. Equation (4*a*) now becomes.

$$G\frac{dr}{r^2} = \frac{d\rho}{\rho_3}\left[\frac{d\left[\frac{\Omega^2}{\theta}\right]}{d\left(\frac{1}{\theta}\right)} - \omega_0^2\,\rho^4\right]$$

$$= \frac{d\rho}{\rho_3}\left[\Omega^2 - \theta\frac{d\Omega^2}{d\theta} - \omega_0^0\,\rho^4\right]$$

In order that the equilibrium may be stable the quantity of contained heat (see equation 4*h*) must increase in the direction towards the celestial pole. But the layers of similar air are less inclined than the inclination of the polar axis at all places where the quantity

$$\Omega^2 - \theta.\ \frac{d\Omega^2}{d\theta} < \omega_0^2\,\rho^4;$$

but on the other hand their inclination is steeper where the left-hand side of this inequality is greater than the right.

## IV. GRADUAL VARIATIONS OF THE EQUILIBRIUM BY FRICTION AND HEATING.

It is well known how very differently the propagation of changes of temperature in the air goes on according as heat is added or withdrawn below or above.

If the lower side of a stratum of air is warmed, as occurs at the surface of the earth, by action of the solar rays, then the heated stratum of air seeks to rise. This is effected very soon all over the surface in small tremulous and flickering streams such as we see over any plane surface strongly heated by the sun; but soon these smaller streams collect into larger ones when the locality affords opportunity, especially on the side of a hill. The propagation of heat goes on relatively rapidly through the whole thickness of the atmospheric layer, and when it has a uniform quantity of heat throughout its whole depth and is therefore in adiabatic equilibrium then also the newly added air seeks *de nova* to distribute itself through the entire depth.

The same process occurs with like rapidity when the upper side of a stratum of air is cooled.

On the other hand, when the upper side is warmed and the lower side cooled such convective movements do not occur. The conduction of heat operates very slowly in large dimensions, as I have already explained above. Radiation can only make itself felt to any considerable extent for those classes of rays that are strongly absorbed. On the other hand, experiments on the radiation from ice and observations of nocturnal frosts show that most rays of even such low temperatures can pass through thick layers of clear atmosphere without material absorption.

Therefore a cold stratum of air can lie for a long time on the earth, or equally a warm stratum remain at an altitude, without changing its temperature otherwise than very slowly.

Similar differences exist also in the case of the change of velocity by friction. For the normal inclination of an atmospheric stratum its upper end is nearer to the earth's axis than its lower end. If the stratum appears at the earth's surface as a west wind, then the moment of rotation of the lowest layer is delayed [by resistance of the earth's surface], its centrifugal force is diminished, and on the polar side of the stratum this lowest portion will slide outwards, approaching the axis in order to find its position of stable equilibrium at the upper end of the stratum. This movement will ordinarily take place in small tremulous streams similar to the ascent of warm air and must diminish the moment of rotation of the whole layer rather uniformly, but in the upper portions a little later than in the lower. Since, however, this latter effect distributes itself throughout the whole mass of air, it will become much less apparent on the lower side of the stratum than if it were confined to the lower stratum.

For the east wind matters are reversed. Its moment of rotation is increased by the friction on the earth's surface. The accelerated mass of air [the ground layer] already finds itself in that position of equilibrium which it has to occupy within its stratum, and can only press forward equatorially along the earth's surface into the stratum lying in front of it. If it is also simultaneously heated then the resulting ascent takes place more slowly than would occur in a stratum of air that is at rest at the bottom.

Hence it is to be concluded that in the east wind, the change due to friction is confined to the lower layer of air, and furthermore that it is relatively more effective here than in the case of a west wind of equal velocity. In general, the retarded layer of air will press forward toward the equator, in the Northern Hemisphere as northeast wind. In this motion it will continue to appear as an easterly wind since it is continually arriving at more rapidly rotating zones on the earth. The air of the stratum lying above the retarded layer will, where the region is free from obstruction, as at the outer border of the trade wind zone, fall behind and will appear as an east wind, retaining its moment of rotation unchanged and gradually pushing toward the equator will itself in its turn experience the above described influence of friction. I would here further remark that the water so abundantly evaporated in the tropical zone also enters into the trade wind, but with the greater velocity of rotation of the revolving earth and must diminish the retardation of the latter with respect to the earth.

The lower layers of the trade wind can press in under the equatorial calm zone itself only when any difference between their velocity of rotation and that of the earth's surface is entirely destroyed. They then blend with the zone of calms and increase its mass so that the latter broadens with its inclined boundary surface always higher above the layer of diminishing east wind beneath it.

Thus it is brought about that whereas below [nearer the earth's surface] mostly continuous changes are taking place in the temperature and the moment of rotation of the strata, on the other hand above, the boundaries of the broadening zones of calms (that have the great moment of rotation that pertains to the equatorial air and which at 10° latitude must appear as a strong west wind, and at 20° latitude as a westerly storm), occur in direct contact with the underlying stratum that has less velocity of rotation and lower temperature. Evidently the upper side of this latter [lower] stratum can scarcely be changed as to the quantity of its contained heat and of its moment of rotation, while after the loss of its lower layer it is being pushed sidewise and towards the equator.

As I have already shown in my communication to this Academy, April 23, 1868, on "Discontinuous Fluid Motions,"* such discontinuous motions can continue for a while, but the equilibrium at their boundary

* [See No. III of this collection of Translations.]

surfaces is unstable, and sooner or later they break up into whirls that lead to general mixture of the two strata. This statement is confirmed by the experiments with sensitive flames and by those in which by means of a cylindrical current of air blown from a tube we make a section in a flame and thus make visible the boundary of the moving and the quiet mass. If, as in our case, the lower stratum is the heavier it can be shown that the perturbations must at first be similar to the waves of water that are excited by the wind. The process is made evident by the striated cirrus clouds that are visible when fog is precipitated at the boundary of the two strata. The great billows of water that are raised by the wind show the same process which is different in degree only, by reason of the greater difference of the specific gravities. The severer storms even turn the aqueous billows to breakers, that is to say, they form caps of froth and throw drops of water from the upper crest high into the air. Up to a certain limit, this process can be mathematically deduced and analyzed, on which subject I propose a later communication. For slighter differences of specific gravity the result of this process must be a mixture of the two strata with a formation of whirls and under some circumstances with heavy rainfall. An observation of one such process under very favorable circumstances I once made accidentally upon the Rigi and have described.*

The mixed strata acquire a temperature and moment of inertia whose values lie between those of the component parts of the mixture, and its position of equilibrium will therefore be found nearer the equator than the position previously occupied by the colder stratum that enters into it. The mixed stratum will descend toward the equator and push back the strata lying on the polar side. Into the empty space thus created above, the strata from which this descending portion has been drawn stretch upwards, and thus their cross section must be diminished. Wherever the lower layers are pushed apart by descending masses of air, as is well known, there arise anti-cyclones; wherever cavities or gaps arise by reason of ascending masses of air, there arise cyclones. Anti-cyclones and the corresponding barometric maxima are shown, with very great regularity, by the meteorological charts † along the very irregularly varying limits of the northeast trade in the Atlantic Ocean—in the winter, under latitude 30°; in summer, under 40° latitude. On account of the inclined position of the strata, the rain that frequently forms by reason of the mixture of air (Dove's Subtropical Rain) falls somewhat farther northward because the water must fall down almost vertically. ‡

---

* See Proceedings of the Physical Society in Berlin, October 22, 1886.

† Daily Synoptic Weather Charts. Published by the Danish Meteorological Institute and the German Seewarte, Copenhagen and Hamburg.

‡ [The results stated in the above paragraph were subsequently greatly modified by Helmholtz. See Section V of his second memoir, or page 98 of these Translations.—*C. A.*]

Therefore the zone of cyclones begins there, but these become more frequent farther northward. We can certainly assume that the process of mixture is not perfected immediately at the exact border of the trade-wind zone, but that a part of the rapidly-rotating warm upper stratum remains unchanged or half mixed, which will presently bring about new mixtures farther on toward the pole.

In general, in this zone of mixture, even below at the earth's surface, the west wind must retain the upper hand because the increase of the total moment of rotation which the mass of air, through friction, experiences in the east wind of the trade zone must finally rise to such a pitch that somewhere the west wind again touches the earth and experiences sufficient friction to entirely give back the increase that it had. The masses of air resting in the equilibrium of stratification can certainly have no long-continued motion of rotation that differs essentially from that of the earth beneath them. When therefore they are mixed with the stronger west wind of the air from above, they receive a movement toward the east. Moreover the falling rain that in great part comes from the upper west winds, must transmit its motion to the lower strata through which the rain falls. Eventually all zones that are pressed polewards by intermixed masses moving equatorially and descending from them will become west winds.

Another permanent source of winds is the cooling of the earth at the poles. The cold layers endeavor to flow outwards from each other at the earth's surface and form east wind (or anti-cyclones). Above these the warmer upper strata must fill the vacancy and continue as west winds (or cyclones). Thus an equilibrium would come about, as is shown in Sect. II, if it were not that the lower cold stratum acquires, through friction, a more rapid movement of rotation, and is therefore competent for further advance. In doing this, according to the above given views this lower stratum must remain on the earth's surface. That in fact it does so is shown by frequent experiences during our northeast winter winds whose low temperatures frequently enough do not extend up to even the summit of the North German Mountains. Moreover on the front border of these east winds advancing into the warmer zone, the same circumstances are effective in order to bring about a discontinuity between the movement of the upper and lower currents, as in the advancing trade-winds, and there is therefore here a new cause for the formation of vortex motions.

The advance of the polar east wind, although recognizable in its principal features, proceeds relatively very irregularly since the cold pole does not agree with the pole of rotation of the earth, and also because low mountain ranges have a large influence. In addition to this comes the consideration that in the cold zone fog causes only a moderate cooling of the thicker stratum of air, but clear air brings about a very intense cooling of the lower layer. By such irregularities, it is brought about that the anti-cyclonic movement of the lower stratum

and the great and gradually increasing cyclone of the upper stratum (that should otherwise be expected at the pole) break up into a large number of irregular, wandering cyclones and anti-cyclones, with a preponderance of the former.

From these considerations, I draw the conclusion that the principal obstacle to the circulation of our atmosphere, which prevents the development of far more violent winds than are actually experienced, is to be found not so much in the friction on the earth's surface as in the mixing of differently moving strata of air by means of whirls that originate in the unrolling of surfaces of discontinuity. In the interior of such whirls the strata of air originally separate are wound in continually more numerous, and therefore also thinner layers spirally about each other, and therefore by means of the enormously extended surfaces of contact there thus becomes possible a more rapid interchange of temperature and equalization of their movement by friction.

The present memoir is intended only to show how by means of continually effective forces, there arises in the atmosphere the formation of surfaces of discontinuity. I propose, at a future time, to present further analytical investigations as to the phenomena of such disturbances of continuity.

VI.

## ON ATMOSPHERIC MOVEMENTS.*

(*SECOND PAPER.*)

By Prof. H. von Helmholtz.

### ON THE THEORY OF WINDS AND WAVES.

In my previous communication made to the Academy on the 31st of May, 1888, I endeavored to prove that conditions must regularly recur in the atmosphere where strata of different density lie contiguous one above another. The reason for the greater density of the lower stratum is conditioned by the fact that the latter has either a smaller amount of heat or a smaller velocity of rotation, if in fact both conditions do not work together. As soon as a lighter fluid lies above a denser one with well-defined boundary, then evidently the conditions exist at this boundary for the origin and regular propagation of waves, such as we are familiar with on the surface of water. This case of waves as ordinarily observed on the boundary surfaces between water and air is only to be distinguished from the system of waves that may exist between different strata of air, in that in the former the difference of density of the two fluids is much greater than in the latter case. It appeared to me of interest to investigate what other differences result from this in the phenomena of air waves and water waves.

It appears to me not doubtful that such systems of waves occur with remarkable frequency at the bounding surfaces of strata of air of different densities, even although in most cases they remain invisible to us. Evidently we see them only when the lower stratum is so nearly saturated with aqueous vapor that the summit of the wave, within which the pressure is less, begins to form a haze. Then there appear streaky, parallel trains of clouds of very different breadths, occasionally stretching over the broad surface of the sky in regular patterns. Moreover it seems to me probable that this which we thus observe under special conditions that have rather the character of exceptional cases, is present in innumerable other cases when we do not see it.

* From the *Sitzungs-berichte* of the Royal Prussian Academy of Sciences at Berlin, July 25, 1889, pp. 761–780.

The calculations performed by me show further that for the observed velocities of the wind there may be formed in the atmosphere not only small waves, but also those whose wave-lengths are many kilometres which, when they approach the earth's surface to within an altitude of one or several kilometres, set the lower strata of air into violent motion and must bring about the so-called gusty weather. The peculiarity of of such weather (as I look at it) consists in this, that gusts of wind often accompanied by rain are repeated at the same place, many times a day, at nearly equal intervals and nearly uniform order of succession.*

I think it may be assumed that this formation of waves in the atmosphere most frequently gives occasion to the mixture of atmospheric strata and, under favorable circumstances, when the ascending masses form mist, give opportunity for disturbances of an equilibrium that had already become nearly unstable. Under conditions, such as those where we see water waves breaking and forming white caps, thorough mixtures must form between the strata of air.

In the beginning of my previous paper I have explained how insufficient are the known intensities of the internal friction and the thermal conductivity of gases in order to explain the equilibration of motions and temperatures in the atmosphere. Since now the mechanical theory of heat has taught us to consider friction in gases as the mixture of strata having different movements, but the conduction of heat as the mixture of strata having different temperatures, it is therefore intelligible that a more thorough mixture of strata in the atmosphere must bring about, to a still higher degree, the effects of friction and conduction,† but certainly not in a quiet, steady progress, but proceeding irregularly as is indeed the special character of meteorological processes.

Therefore I have considered it important to develop the theory of waves at the common boundary surface of two fluids. Hitherto in studies on waves of water, so far as known to me, the influence of the air and its motion with the water has always been neglected, but this may not be done in the present work. The problem becomes thereby much more complicated and difficult; and since even the simpler problem that takes no account of the influence of the wind has at the hands of many excellent mathematicians received only incomplete and approximate solutions, under assumptions chosen to simplify the problem, therefore I pray to be excused in that I also have at first treated the simplest case of the problem, namely, the movement of rectilinear waves which propagate themselves with unchanged forms

---

* This assumption of the formation of billows in the atmosphere that I recently briefly expressed in my first contribution has since then also been propounded by Jean Luvini (*La Lumière Électrique*. T. xxx, pp. 368, 617, 620).

† Perhaps this would correspond to the assumptions that form the basis of the theory submitted to this (Berlin) Academy by Oberbeck, March 15, 1888. [See Nos. XII and XIII of this collection.—*C. A.*]

and with uniform velocity on the plane boundary surface between indefinitely extended layers of two fluids of different densities and having different progressive movements. I shall call this kind of billows stationary billows, since they represent a stationary motion of two fluids when they are referred to a system of coördinates which itself advances with the waves. Since in the relative motion of the different parts of a closed material system nothing is changed when the whole receives a uniform rectilinear velocity toward any direction, therefore this rearrangement of our problem is allowable.

Moreover I propose to-day to give only the results of my mathematical investigations. The complete presentation of these I reserve for publication in another manner.

Before I advance to the theory of atmospheric billows, I will however introduce a supplement to the considerations given in my communication of May, 1888, by which the region in which we have to look for the conditions that give rise to atmospheric billows is better defined.

## V. THE ASCENT OF MIXED STRATA.

In Section III of my previous communication I have shown what would be the law of equilibrium, in case such a condition should be attained, between atmospheric rings of different temperatures and different speeds of rotation, which however are all assumed as being composed of mixtures that are similar to each other. I now return to equation (4*a*, page 85). Let the location of a point in the atmosphere be given by the quantities

$\rho$, the distance from the earth's axis.

$r$, the distance from the center of the earth.

Let $\omega_0$ be the angular velocity of the solid earth; and $\Omega_1$ and $\Omega_2$ be the constant moments of rotation of the unit of mass of one or the other layer of air:

Let $\theta_1$ and $\theta_2$ be the quantities that I have called the contained caloric of the unit of mass of air, and that certainly may be better designated by the term *potential temperatures*, so well chosen by Bezold, namely, those temperatures which the respective masses of air would assume when brought adiabatically to the normal pressure.

Let $G$ = constant of gravitation. In accordance with equation (4*a*) we now have at the boundary surfaces the relation

$$\frac{G}{r^2}dr = \frac{d\rho}{\rho^3}\left[\frac{\Omega_1^2\,\theta_2 - \Omega_2^2\,\theta_1}{\theta_2 - \theta_1} - \omega_0^2\,\rho^4\right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The ratio $\frac{d\rho}{dr}$ indicates also the ratio of the sines of the two angles which the tangent to the curve in the meridional plane makes on the one hand with the earth's axis, and on the other hand with the horizon. When, as is ordinarily the case, the warmer layer has also the greater

moment of rotation, the ratio $\frac{d\rho}{dr}$ is then negative, and the tangent to the boundary surface cuts the celestial vault below the pole. The colder, more slowly rotating mass, which we will designate by the subscript (2), lies in the acute angle between the boundary surface and that part of the terrestrial surface which is on the polar side of the given point.

When now at the boundary surface of the two strata, a mixture takes place of the component masses $m_1$ and $m_2$, then will the moment of rotation ($\Omega$) of the mixed masses be given by the equation

$$(m_1+m_2)\Omega=m_1\Omega_1+m_2\Omega_2,$$

since the sum of the moments of rotation does not vary when no exterior rotatory forces are at work. Equally will the potential temperature $\theta$ of the mixture be given by

$$(m_1+m_2)\ \theta=m_1\ \theta_1+m_2\ \theta_2.$$

If now in equation (1), we at first substitute the mixture in place of the cooler mass (2), in order to find the direction of the boundary line between the mass (1) and the mixture, and indicate by $d\rho_1$ and $dr_1$ the corresponding values of $d\rho$ and $dr$, then our equation (1), after an easy transformation, gives

$$\rho^3\frac{G}{r^2}\left[\frac{dr_1}{d\rho_1}-\frac{dr}{d\rho}\right]=\frac{m_1\ \theta_1}{m_1+m_2}\ \frac{(\Omega_1-\Omega_2)^2}{\theta_2-\theta_1} \quad . \quad . \quad . \quad . \quad (1a)$$

Since in stable equilibrium $\theta_2<\theta_1$, therefore this equation shows that

$$\frac{dr_1}{d\rho_1}<\frac{dr}{d\rho} \text{ or } \frac{d\rho_1}{dr_1}>\frac{d\rho}{dr},$$

that is to say, that the boundary surfaces between mass (1) and the mixture must ascend more steeply with reference to the horizon than the boundary surface between (1) and (2).

Similarly it follows that the ratio $\frac{dr_2}{d\rho_2}$ between the cooler mass (2) and the mixture will be given by the equation—

$$\rho^3\frac{G}{r^2}\left[\frac{dr_2}{d\rho_2}-\frac{dr}{d\rho}\right]=\frac{m_2\theta_2}{m_1+m_2}\ \frac{(\Omega_1-\Omega_2)^2}{\theta_1-\theta_2}.$$

Therefore $\frac{dr_2}{d\rho_2}>\frac{dr}{d\rho}$; that is to say, the boundary surface between the cooler mass (2) and the mixture must make a more acute angle with the horizon towards the pole than does the boundary surface between the mixture and the warmer mass (1).

It is to be noted that the ratios $\frac{d\rho}{dr}$ are positive when the tangent to the boundary line is more inclined than the line to the pole—in the other cases they are negative—and furthermore that the increase of a negative quantity means the diminution of its absolute value.

But the required directions for the two boundary lines of the mixture can only exist when this mixture passes upwards between the two masses (1) and (2). Only thus can there be a condition of equilibrium.

Hence results the important consequence that all newly formed mixtures of strata that were in equilibrium with each other must rise upwards between the two layers originally present, a process that of course goes on more energetically when precipitations are formed in the ascending masses.

While the mixed strata are ascending, those parts of the strata on the north and south that have hitherto rested quietly approach each other until they even come in contact, by which motion the difference of their velocities must necessarily increase since the strata lying on the equatorial side acquire greater moment of rotation with smaller radius, while those on the polar side acquire feebler rotation with a larger radius. If this occurs uniformly along an entire parallel of latitude we should again obtain a new surface of separation for strata of different rates of rotation whose equatorial side would show stronger west winds than the polar side, which latter might occasionally show east winds. On account of the numerous local disturbances of the great atmospheric currents there will, as a rule, be formed no continuous line of separation, but this will be broken into separate pieces which must appear as cyclones.

But as soon as the total mixed masses have found their equilibrium the surfaces of separation will again begin to form below, and new wave formations will initiate a repetition of the same processes.*

From these considerations it follows that the locality for the formation of billows between the strata of air is to be sought especially in the lower parts of the atmosphere, while in the upper parts an almost continuous variation through the different values of rotation and temperature is to be expected. The boundary surfaces of different strata of air, along which the waves travel, have one edge at the earth's surface and there the strata becomes superficial. Experience also teaches, as does the theory, that water-waves that run against a shallow shore break upon it, and even waves which originally run parallel to the shore propagate themselves more slowly in shallow water. Therefore waves that are originally rectilinear and run parallel to the banks will

---

* In the last section of my previous paper [see *ante* p. 91] I located the origin of the discontinuity principally in the upper strata of the atmosphere. But in that paper the point of departure was different from the present. In that the question considered was: If at any time the atmosphere has attained an initial stage of continuous steady motion without surfaces of separation, where will such a surface first form? To this the answer is: At the upper boundary of the tropical belt of calms.

At present the question is, Where in consequence of processes of mixture will the surfaces of separation necessarily be renewed? But I must take back the proposition on page 91 that treats of the descent of mixed strata, now that I have found the law expressed in this paragraph.

in consequence of the delay become curved, whereby the convexity of their arcs is turned toward the shore; in consequence of this they run upon the shore and break to pieces there.

In the next paragraphs I will show in what respects the movements and forms of water-waves must be changed in order to be applicable to the air. These relations are indeed not to be rigidly transferred from water-waves that break upon the shore to the air, and even the simpler theory hitherto developed, which neglects the influence of the air, gives no complete explanation on this point.

But the conditions are not very different from those cases in which we can make a strict application, and I therefore believe there is no reason to doubt that waves of air which in the ideal atmospheric circulation symmetrical to the axis could only progress in a west-east direction, must, when once they are initiated in the real atmosphere, turn down toward the earth's surface and break up by running along this in a northwesterly direction (in the northern hemisphere).

Another process that can cause the foaming of the waves at their summits is the general increase in velocity of the wind. My analysis also demonstrates this: it shows that waves of given wave-length can only co-exist with winds of definite strength. An increase in the differential velocities within the atmosphere indeed often happens, but one can not yet give the conditions generally effective for such a process.

I will here also mention another point that may give rise to considerations against my explanation. Water-waves forced up to a great height always have narrow, strongly curved ridges and broad, flat, curved troughs. Analysis shows that this feature is independent of the nature of the medium. Atmospheric waves have, on the other hand, rounded heads when they become visible to us as bands of cirri. But we must remember that according to the proposition first formulated by Reye, air that has formed cloud or mist is lighter than it was before. Therefore what we see as mist rises up and increases the size of the summit of the wave more than would be the case in transparent air.

## VI. CONSEQUENCES DEDUCED FROM THE PRINCIPLE OF MECHANICAL SIMILARITY.

If we confine ourselves to the search for such rectilinear waves as advance with uniform velocity without change of form, we may, as before remarked, represent such a movement as a stationary one, by attributing to both the media a uniform rectilinear velocity equal and opposite to that of the wave. It is well known no change is thereby introduced into the relative motions of the different parts of the masses. In this way the bounding surface of the two media appears as a surface fixed in space; above it the upper medium flows in one direction; below it the other medium in the opposite direction. At a great distance from the bounding surface both movements become rectilinear

currents of uniform velocity, but in the neighborhood of the wavy boundary surface the motion must follow its direction.

Designate by $u$ and $v$ the components of the velocities of the fluid particles at the point corresponding to the rectangular coördinates $x$ and $y$; these velocities are by assumption, independent of the time, and (for an incompressible fluid whose current is free from vortices) can be presented in the form

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x},$$

where $\psi$ is such a function of the coördinate as satisfies the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equations

$$\psi = \mathrm{const}$$

are in this case, as is well known, the stream-lines of the fluid. The boundary line of both fluids must be such a stream-line, and we will give it for both sides the value

$$\bar{\psi}_1 = 0 \text{ and } \bar{\psi}_2 = 0.$$

The above overscored letters will, in what follows, always indicate values on the boundary surface.

The first boundary condition that we have to satisfy is therefore that, when we express $\psi_1$ and $\psi_2$ as functions of $x$ and $y$, then the two equations

$$\bar{\psi}_1 = 0 = \bar{\psi}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

shall admit of an accordant solution.

The second boundary condition is that the pressure at the bounding surface shall be the same on both sides, or

$$\bar{p}_1 = \bar{p}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

Now, under the adopted assumptions and when $s$ is the density of the fluid and $C$ is a constant, we have

$$p = C - s\, g\, x - \tfrac{1}{2} s \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right].$$

Therefore the equation (2b) can be written:

$$\text{Const.} = (s_1 - s_2)\, g\, \bar{x} + \tfrac{1}{2} s_1 \left( \frac{\partial \bar{\psi}_1}{\partial N_1} \right)^2 - \tfrac{1}{2} s_2 \left( \frac{\partial \bar{\psi}_2}{\partial N_2} \right)^2 \quad . \quad . \quad . \quad . \quad (3)$$

The equations (2) and (2$a$) remain true when we increase either the values of the two coördinates $x$ and $y$ or those of $\psi_1$ or $\psi_2$ in any given ratio. Since the densities $s_1$ and $s_2$ do not occur in these two equations, therefore also these can change to any amount. But equation (3) requires that the quantities

$$\frac{s_1}{s_2-s_1}\left(\frac{\partial\psi_1}{\partial N_1}\right)^2\frac{1}{x} \text{ and } \frac{s_2}{s_2-s_1}\left(\frac{\partial\psi_2}{\partial N_2}\right)^2\frac{1}{x}$$

shall remain unchanged. When therefore $s_1$ and $s_2$ vary and we put their ratio

$$\frac{s_1}{s_2}=\sigma$$

and when further the coördinates increase by the factor $n$, but $\psi_1$, by the factor $a_1$ and $\psi_2$ by the factor $a_2$, then the quantities

$$\frac{\sigma}{1-\sigma}\cdot\frac{a_1^2}{n^3} \text{ and } \frac{1}{1-\sigma}\cdot\frac{a_2^2}{n^3}$$

must both remain unchanged.

Or when we, in the expressions for these quantities, put

$$b_1=\frac{a_1}{n} \text{ and } b_2=\frac{a_2}{n}$$

as the ratios by which the velocities are altered, then the above proposition becomes equivalent to saying that the geometrically similar waveforms can occur when

$$\frac{\sigma}{1-\sigma}\cdot\frac{b_1^2}{n} \text{ and } \frac{1}{1-\sigma}\cdot\frac{b_2^2}{n}$$

remain unchanged.

(1) *If the ratios of the densities are not changed then in geometrically similar waves, the linear dimensions increase as the squares of the velocities of the two media; the velocities therefore will increase in equal ratios.*

Therefore for a doubled velocity of the wind we shall have waves of four times the linear dimensions.

This proposition is not limited to stationary movements, but is quite general.* The following propositions however will hold good only for stationary waves.

(2) *When the ratio of the density $\sigma$ is varied, the quantities*

$$\sigma\frac{b_1^2}{b_2^2}=\frac{s_1}{s_2}\frac{b_1^2}{b_2^2}=\text{const.}$$

*See my paper "On a Theorem relative to geometrically similar movements of Fluid Bodies," in the *Monats b. der Akad. Berlin*, 1873, pages 501 to 514; [or see No. IV of this collection of Translations.]

*must remain constant;* that is to say, *the ratio of the living forces of the corresponding units of volume must remain unchanged.* As corresponding units of volume, those must be used that hold good in the region of rectilinear flow far from that of the wave surface; but also for such units of volume as have centers that are corresponding images for each other the same proposition holds good.

(3) If for a varied density the geometrically similar waves are to have the same wave-length, namely, $n=1$, then

$$b_1 \text{ must increase as } \sqrt{\frac{1}{\sigma}-1}=\sqrt{\frac{s_2-s_1}{s_1}}$$

$$b_2 \text{ must increase as } \sqrt{1-\sigma}=\sqrt{\frac{s_2-s_1}{s_2}}.$$

For air and water at a temperature of 0° C. we have the ratio

$$\sigma=\frac{1}{773.4}$$

For two strata of air whose temperatures are 0° and 10° the ratio becomes

$$\sigma=\frac{273}{283}$$

If both boundary surfaces are to show congruent waves and therefore also equal wave-lengths, and if I designate by $\beta_1$ and $\beta_2$ the values of the quantities $b_1$ and $b_2$ in this last case, then we have

$$b_1=145.21\beta_1$$
$$b_2=5.316\beta_2$$

therefore both the velocities, especially that of the wind relative to the waves of water, must be considerably diminished for the case of ærial billows.

The value of the quantity

$$\mathfrak{p}=\frac{s_2 \, . \, b_2^2}{s_1 \, . \, b_1^2}$$

which is invariable for any change in the material for a given form of wave whose store of energy is equal to that of the rectilinear flow along a plane boundary surface is given at least approximately according to my computations, as

$$\mathfrak{p}=0.43103.$$

If by a wind-force $w$ we understand the difference of the movement of the two media

$$w=b_1+b_2$$

then will for air and water

$$\frac{b_2}{w}=0.069469$$

and if $w=10\ \frac{\text{metres}}{\text{second}}$

$$\lambda=0.208965 \text{ metre}$$

on the other hand for the two strata of air

$$\frac{\beta_2}{\beta_1+\beta_2}=0.67135$$

and for $w=10\ \frac{\text{metres}}{\text{second}}$

$$\lambda=549^{\text{m}}.65$$

Hence it results that when we would obtain for this form of atmospheric wave the same wind velocity as for geometrically similar water-waves we must increase the wave-length of the air wave in the ratio of 1 to 2630.3.

This ratio becomes somewhat smaller when we execute the computation for the lowest waves for which

$$\mathfrak{p}=0.15692$$

This gives for air and water

$$\frac{b_2}{w}=0.090776$$

and for a wind velocity of 10 metres per second,

$$\lambda=0.^{\text{m}}83222$$

The necessary magnification of the wave-length for equal strength of wind would be 1:2039.6 which gives a wave-length of more than 900 metres for a wind of 10 metres per second.

Since the moderate winds that occur on the surface of the earth, often cause water-waves of a metre in length, therefore the same winds acting upon strata of air of 10° difference in temperature, maintain waves of from 2 to 5 kilometres in length. Larger ocean-waves from 5 to 10*m* long would correspond to atmospheric-waves of from 15 to 30 kilometres, such as would cover the whole sky of the observer and would have the ground at a depth below them less than that of one wave-length, therefore comparable with the waves in shallow water, such as set the water in motion to its very bottom.

The principle of mechanical similarity, on which the propositions of this paragraph are founded, holds good for all waves that progress with an unchanged form and constant velocity of progress. Therefore these propositions can be applied to waves in shallow water, of uniform

depth, provided that the depth of the lower stratum in the image varies in the same ratio as the remaining linear dimensions of the waves.

The velocity of propagation of such waves in shallow water depends on the depth of the water. For water waves of slight height and without wind it can be computed by well-known formulæ. When we indicate the depth of the water by $h$ and put $n=\frac{2\pi}{\lambda}$, then is

$$b^2=\frac{g}{n}\cdot\frac{e^{nh}-e^{-nh}}{e^{nh}+e^{-nh}}$$

which for $h=\infty$ becomes

$$b^2=\frac{g}{n}=\frac{g\lambda}{2\pi}$$

and for small values of $h$ becomes

$$b^2=gh$$

When however the depth of the water is not small relatively to the wave length, then the retardation is unimportant, thus for

$\frac{h}{\lambda}=\frac{1}{2}$ the speed of propagation diminishes as 1:0.95768

$=\frac{1}{4}$ the speed of propagation diminishes as 1:0.80978

$=\frac{1}{10}$ the speed of propagation diminishes as 1:0.39427

When it is calm at the earth's surface the wind beneath the trough of the aerial billow is opposed to the direction of propagation, but under the summit of the billow it has the same direction as that. Since the amplitudes at the earth's surface are diminished in the proportion $e^{-nh}:1$ with respect to the amplitudes at the upper surface, therefore these latter variations can only make themselves felt below when the depth is notably smaller than the wave-length. Variations of barometric pressure are only to be expected when decided changes in the wind are noticed during the transit of the wave.

### VII. FUNDAMENTAL FORMULÆ FOR THE COMPUTATION.

I will here give the theory of the calculation only so far as is necessary, so that any investigator familiar with analytical methods can verify my results. I introduce two new variables, $\eta$ and $\theta$, which are so connected with rectangular coördinates $x$ and $y$ that

$$e^{n(x+yi)}=a[\cos(\theta+\eta i)-\cos\varepsilon] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

wherein $n$, $a$, and $\varepsilon$ are constants. The boundary line between the two fluids corresponds to a constant positive value of $\eta$, namely:

$$\eta=h$$

Hence for this boundary line result the equations

$$\left.\begin{aligned} e^{nx} \cos (ny) &= a(\cos ih \cos \theta - \cos \varepsilon) \\ e^{nx} \sin (ny) &= -\frac{a}{i} \sin (ih) \sin \theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1a)$$

By the elimination of $\theta$ this gives an equation between $x$ and $y$ as the equation of the boundary line. Beside the constant $a$ which determines the initial point of the $x$ coördinate and the $n$ which determines the wave-length this equation contains two arbitary parameters $h$ and $\varepsilon$ that determine the form of the curve.

We take $x$ vertical, increasing upwards, and then for the space occupied by the upper fluid, for which we use the subscript $_1$ put

$$\psi_1 + \varphi_1 i = b_1(\eta - h - i\theta)$$

by which $\psi + \varphi\, i$ becomes simultaneously a function of $(x+yi)$. When $h=\eta$, then $\psi_1=0$, so the boundary line on the lower side coincides with the stream line. When $\eta=+\infty$ then

$$n(x+yi) = \eta - i\theta = \frac{1}{b_1}[\psi_1 + \varphi_1 i] + h$$

or

$$\psi_1 = nb_1 x,$$
$$\varphi_1 = nb_1 y,$$

so that at great altitudes the motion is a rectilinear flow with the velocity $nb_1$.

For the lower space where $\eta < h$ and $x$ has generally a negative value, I put

$$\frac{1}{b_2}[\psi_2 + \varphi_2 i] =$$

$$-nx - nyi + \log\left(\frac{a}{2}\right) + h - 2\sum_{\mathfrak{a}=1}^{\mathfrak{a}=\infty}\left[\frac{1}{\mathfrak{a}}\cdot e^{-\mathfrak{a}h}\cdot\frac{\cos(\varepsilon\mathfrak{a})\,.\cos\mathfrak{a}\,(\theta+\eta i)}{\cos(\mathfrak{a}hi)}\right].$$

Hence for $\eta=h$ there results

$$\frac{1}{b_2}\psi_2 = -nx + \log\left(\frac{a}{2}\right) + h - 2\sum_{1}^{\infty}\left[\frac{1}{\mathfrak{a}}\cdot e^{-\mathfrak{a}h}\cdot \cos(\varepsilon\mathfrak{a}) \cos \mathfrak{a}\theta\right].$$

When we determine the value of $x$ from the equation (1) it is seen that for $\eta=h$ there results $\psi_2=0$, therefore it is seen that the boundary line for the second medium is also a stream-line.

According to equation (1) for $x=-\infty$ we have

$$\cos\theta\,.\cos\eta i = \cos\varepsilon$$

$$\sin\theta\,.\sin\eta i = 0$$

The values corresponding to these are

$$\sin \eta i = 0$$

$$\cos \theta = \cos \varepsilon$$

In consequence of this the equation above given becomes

$$\frac{1}{b_2} \cdot \psi_2 = -nx + \log\left(\frac{a}{2}\right) + h - 2 \sum_1^\infty \left[\frac{e^{-ah} \cdot \cos^2(\varepsilon a)}{a \cdot \cos(ahi)}\right]$$

$$(x = -\infty)$$

The first term of the right-hand member is infinite, but all the others finite when $h$ is a positive quantity. Therefore at great depths the value of $\psi_2$ reduces to

$$\psi_2 = -nb_2x$$

that is to say that even there also the motion is a rectilinear flow with the velocity $-nb_2$.

The second boundary condition which has respect to the equality of pressure on both sides of the boundary surface can, however, by reason of the assumptions already made, be satisfied only approximately for waves of small altitude. The convergence of the series under consideration in this case depends upon the factor $e^{-ah}$. When the quantity $h$ is positive and not too small the series converges relatively rapidly and we obtain for this case sufficient approximation to the true value, in that in the value of the pressure as deduced from equation (3) we equate to zero the terms multiplied by the first to the third power of $e^{-h}$, or of $\frac{1}{\cos hi}$. The terms that do not contain these factors serve only to determine the value of the constant of integration which forms the left-hand side of the equation. These terms just mentioned are linear functions of $\cos \theta$, $\cos 2\theta$, $\cos 3\theta$, and by equating to zero the coefficients of these three quantities we satisfy equation (3) to terms that contain the fourth or higher power of $\frac{1}{\cos hi}$. But this assumption corresponds only to a single possible form of wave, not to the most general form. It has been chosen as an example on account of the simplicity of computation. The three equations that we obtain in this manner are those given below. For brevity we have put

$$\mathfrak{P} = \frac{s_1 \cdot b_1^2 \cdot \pi}{g \cdot \lambda \cdot (s_2 - s_1)}$$

$$\mathfrak{Q} = \frac{s_2 \, b_2^2 \cdot \pi}{g \cdot \lambda (s_2 - s_1)}$$

$$\zeta = \frac{1}{\cos hi}$$

$$z = \zeta \cos \varepsilon$$

The quantity $z$ determines the altitude of the wave, which according to equation (1$a$) is—

$$H = \frac{\lambda}{2\pi} . \log \text{nat} . \left( \frac{1+z}{1-z} \right).$$

The three equations referred to may now be written:

$$\text{I.} \quad z \left\{ \mathfrak{Q} \left[ 2 - 2z^2 + \tfrac{1}{2}\zeta^2 \right] + \mathfrak{P} \left[ 2 + \tfrac{3}{2}\zeta^2 \right] - (1 - \zeta^2) \right\} = 0.$$

$$\text{II.} \quad \mathfrak{Q} \left[ 2z^2 - \zeta^2 \right] - \mathfrak{P} . \zeta^2 - \tfrac{1}{2}z^2 + \tfrac{1}{4}\zeta^2 = 0.$$

$$\text{III.} \quad z \left\{ \mathfrak{Q} \left[ 2z^2 - \tfrac{3}{2}\zeta^2 \right] + \mathfrak{P} . \frac{\zeta^2}{2} - \tfrac{1}{3}z^2 + \tfrac{1}{4}\zeta^2 \right\} = 0.$$

Of the four quantities that occur herein any three may therefore in general be determined by the fourth. Only one system of values, namely,

$$z = 0 \text{ and } \mathfrak{Q} + \mathfrak{P} = \tfrac{1}{4},$$

leaves $\zeta$ undetermined. This solution holds good for the entire lower wave, for which $z$ is to be neglected as compared with $\zeta$.

Since in general one of the four quantities in the equations I to III remains undetermined, therefore for given properties of the medium and for a given strength of the wind, there remains always one variable parameter of the stationary wave; and in fact the further investigation shows that this variable is connected with the quantity of energy that is accumulated in the wave.

The simplest method of computation is to express the remaining quantities as functions of the cos $\varepsilon$.

$$\mathfrak{Q} = \frac{7}{36} . \frac{\cos^2 \varepsilon - \frac{9}{14}}{\cos^2 \varepsilon - \frac{2}{3}}$$

$$\mathfrak{P} = -\tfrac{1}{9} \cos^2 \varepsilon + \tfrac{1}{3} \mathfrak{Q} = -\tfrac{1}{9} \frac{(\cos^2 \varepsilon - \frac{1}{2}).(\cos^2 \varepsilon - \frac{3}{4})}{\cos^2 \varepsilon - \frac{2}{3}}$$

$$\zeta^2 [\mathfrak{Q} (\cos^2 \varepsilon - \tfrac{1}{4}) - \tfrac{3}{4} \mathfrak{P} + \tfrac{1}{2}] = \mathfrak{Q} + \mathfrak{P} - \tfrac{1}{2}$$

Since $\mathfrak{Q}$ and $\mathfrak{P}$ must necessarily be positive, it follows from the first of these equations that

$$\cos^2 \varepsilon > \tfrac{2}{3} = 0.666667;$$

or,

$$\cos^2 \varepsilon < \tfrac{9}{14} = 0.642857.$$

The equation for $\mathfrak{P}$ would also allow $\cos^2 \varepsilon > \frac{2}{3}$, but

$$0.5 < \cos^2 \varepsilon < 0.642857.$$

Finally the equation for $\zeta^2$ can be written

$$\zeta^2 = 0.4 \times \frac{(0.68615 - \cos^2 \varepsilon)\ (\cos^2 \varepsilon + 2.18615)}{(\cos^2 \varepsilon - 0.66537)\ (\cos^2 \varepsilon + 1.46537)}.$$

Since $\zeta^2$ must be positive it follows that

$$0.66537 < \cos^2 \varepsilon < 0.68615;$$

so that values of $\cos^2 \varepsilon$ that are smaller than 0.643 are thereby excluded. But when we consider that for values of $\zeta$ that are larger than 1, the above-given series for the coördinates of the boundary surface are no longer convergent, there results a lower limit that is still higher than the preceding, which corresponds to the value

$$\cos^2 \varepsilon > 0.67264 = -\tfrac{1}{2} + \sqrt{\tfrac{11}{8}}.$$

For this value the altitude of the wave will still be finite, namely:

$$H = \frac{\lambda}{2\pi} \times 2.5112 = \lambda\, 0.39967.$$

But the fact that the value of the coördinates can no longer be developed in converging series, according to the powers of cos $(\alpha\theta)$ and sine $(\alpha\theta)$, shows that a discontinuity or an ambiguity of the coördinates must have come into existence. In fact the equations (1a) also show that for small values of $h$

$$\operatorname{tang}(n\, y) = -\frac{h \sin \theta}{\cos \theta - \cos \varepsilon}$$

$$c^{2nx} = a^2(\cos \theta - \cos \varepsilon)^2.$$

From the first of these it follows that wherever tan $(n\, y)$ has a finite value then cos $\theta$ must be nearly equal to cos $\varepsilon$, and only at the points where tan $(n\, y)$ is very small and passes through zero can $\theta$ increase and rapidly pass through the interval to the next point, where cos $\theta$ approaches again the value, cos $\varepsilon$.

Now for such values of $h$ the diminution of the terms in the series expressing the value of the pressure will not be rapid enough, in order to express the value of the function sufficiently well by using only the first three terms of the series, and the true form of the wave curve for such values of $h$ can only be obtained by further approximations. However, these relations show that waves which rise too high lose the continuity of their surfaces. But sharp ridges can not occur on the surfaces of the waves except when they are at rest relatively to the medium into which they protrude. For when the medium flows around the edge there would occur infinite velocity and infinite pressure at the place in question, which must violently draw up the other liquid, as in fact is occasionally observed in high and foaming waves.

In the case of waves that advance with the same velocity as the wind the summits can in fact have a ridge of 120° before they break into foam.

The above given formulæ show that when cos $\varepsilon$ diminishes from its upper to its lower value, then both $\mathfrak{Q}$ and $\mathfrak{P}$ and $\zeta^2$ must continually increase. For waves whose lengths remain constant the increase of $\mathfrak{P}$ and $\mathfrak{Q}$ means an increase of the two velocities $b_1$ and $b_2$ as well as their sum, *i. e.*, the wind velocity $w=b_1+b_2$. If the latter remains constant, then the wave length must necessarily diminish with increasing cos $\varepsilon$.

It follows from this, that within certain limits the same wind can excite this form of waves of greater and smaller wave lengths. The longer waves will at the same time have a relatively greater altitude. This relation depends upon the store of energy that is accumulated in the wave.

## VIII. THE ENERGY OF THE WAVES.

When we investigate the energy of the waves of water raised by the influence of the wind, and compare it with that which would be appropriate to the two fluids uniformly flowing with the same velocity when the boundary surface is a plane, we find that a large number of the possible forms of stationary wave motion demand a smaller storage of energy than the corresponding current with a plane boundary. Hence the current with a plane boundary surface plays the part of a condition of unstable equilibrium to the above-described wave motion. Besides these, there are other forms of stationary wave motion where the store of energy for both the masses that are in undulating motion is the same, as in the case of currents of equal strength with plane bounding surfaces; and finally, there are those in which the energy of the wave is the greater.

The reason for this is to be found in the following circumstances: In the undulating masses of water two forms of energy occur, namely:

*First*, potential energy, represented by the water raised from the wave valley to the wave summit. This quantity of work increases with the increasing height of the wave, and must always be positive; it is only absent for perfectly smooth surfaces.

*Second*, living force is common to the two forms of motion under comparison, and according to the original assumption there is an equal quantity of it in the portions of the fluid masses distant from the boundary surface. The difference of the two modes of motion is not affected by the participation of the more distant strata of fluid, the difference between the two motions depends only on the strata that lie near the boundary surface. The wave surface which we again imagine to ourselves fixed in space affords to the two fluids streaming along it an alternately broad and narrow channel; where the bed is broader the fluid moves more slowly, the upper fluid above the wave valley, the lower fluid under the wave summit. Thereby the living force of the portion flowing through a broadening of the channel will be alternately smaller, while that flowing through a narrowing of the channel will be greater than the living force in the corresponding part of the uniform

stream with the plane bounding surface. But the volumetric extension of the part with diminished living force, that fills the broader channel, is greater than the volume of increased velocity in the narrow channel. Therefore in the sum total the living force of the diminished portion prevails.

Nevertheless only the terms of the fourth degree in $\zeta$ which first occurred in the computation by considering the terms with $\zeta^3$ in the values of $x$ and $y$, give a basis for the computation of the difference of energy. This difference, as computed for one wave-length according to my calculation in the class of waves discussed in Section VII, is as follows:

$$E\frac{1}{2\pi g(s-s_0)}=\mathfrak{Q}\cdot\frac{z^2}{4}\left[5\zeta^2-2z^2\right]+\frac{1}{48}\left[z^4-15z^2\zeta^2-\frac{3}{4}\zeta^4\right]$$

or

$$E\frac{1}{2\pi g(s_2-s_1)}=$$

$$\frac{7\cos^2\varepsilon}{144}\cdot\zeta^4\frac{[5-2\cos^2\varepsilon]\,[\cos^2\varepsilon-\frac{9}{14}]}{\cos^2\varepsilon-\frac{2}{3}}-\frac{1}{48}\zeta^4[15.0845-\cos^2\varepsilon].[\cos^2\varepsilon+0.0845]$$

In this the $\mathfrak{Q}$ is the only factor that changes rapidly for small changes of $\cos\varepsilon$, a circumstance that very materially lightens the numerical computation. For $E=0$, we find the value $\cos^2\varepsilon=0.675148$, which is not very far from the limit of convergency or $\cos\varepsilon^2=0.67264$.

Corresponding to $E=0$ we find

$$\begin{aligned}\mathfrak{Q}&=0.740333\\ \mathfrak{P}&=0.1717613\\ \zeta&=0.6899\\ z&=0.56686\\ H&=0.20464\times\lambda\\ e^h&=2.52006\end{aligned}$$

Since these are the waves that can be immediately produced by a constant wind, therefore these are the values that lie at the foundation of the computation quoted in Article 6, whereas the values for the lowest waves are found when we assume for $\cos^2\varepsilon$ the upper limit of its values, namely, 0.68615.

Theory shows, moreover, as also the above numerical example, that the waves of this form for large values of $\cos\varepsilon$ and for the same material and same strength of wind have greater wave-lengths; that, however, their altitudes form a smaller fraction of the wave-length, and that their energy when $\cos^2\varepsilon > 0.675148$ is smaller than that of the rectilinear flow of both media with the same velocities. The difference of energy is zero for very low waves; it is negative when we pass to relatively high waves; it reaches a maximum, then diminishes, and is again zero for the given boundary value.

It is sufficient to have proven that for one form of wave billows due to wind are possible, which billows have a less store of energy than the same wind would have over a plane boundary surface. Hence it follows that the condition of rectilinear flow with plane boundary surface appears at first as a *condition of indifferent or neutral equilibrium*, when we consider only the lower powers of small quantities. But if we consider the terms of higher degree, then this condition is one *of unstable equilibrium*, in view of certain disturbances that correspond to stationary waves between definite limits as to wave-length; but on the other hand is a condition of *stable equilibrium* when we consider shorter waves.

This result is evidently of great importance for the origin of waves. It follows from this, as we everywhere see confirmed in nature, that even the most uniform wind can not blow over a plane surface of water without on the slightest disturbance causing waves of a certain length, which for a given height acquire regular form and speed of propagation. If the wind increases then the heights of all these waves increase, the shorter ones among them break foaming, so that new longer ones of less height can be formed.

The greater energy that is necessary in this case in order to push the shorter waves up higher becomes possible in that the previous feebler wind had already given a part of its energy to the mass of water, and the new stronger wind finds this part already present there.

Breaking foaming atmospheric billows cause mixture of strata in the mass of air. Since the elevations of the air-waves in the atmosphere can amount to many hundred metres, therefore precipitation can often occur in them which then itself causes more rapid and higher ascent. Waves of smaller and smallest wave-length are theoretically possible. But it is to be considered that perfectly sharp limits between atmospheric strata having different motions certainly seldom occur, and therefore in by far the greater number of cases only those waves will develop whose wave-length is very long compared with the thickness of the layer of transition.

The circumstance that the same wind can excite waves of different lengths and velocities, will cause interferences to occur between the waves, and also higher and lower wave summits to follow each other interchangeably. This is a process observed often enough on the shore of the ocean. But where two wave summits of different groups of waves reënforce each other a height will easily be attained at which they break into foam, and thereby, as in the analogous case of the production of sonorous combination tones, longer waves can be formed which, when they are favored by the strength of the wind, can also grow larger. This is one of the processes by which waves of great length can arise.

VII.

## THE ENERGY OF THE BILLOWS AND THE WIND.*

By Prof. H. Von Helmholtz.

In my communication to the Academy on July 25, 1889, I called attention to the fact that a plane surface of water above which a steady wind is blowing is in a state of unstable equilibrium and that the origin of large waves or billows of water is essentially due to this circumstance. I have there also shown that the same process must be repeated at the boundary of two strata of air of different densities gliding over each other, but that in this case it can assume much larger dimensions and without doubt has an important meaning as a cause of nonperiodic meteorological phenomena.

The importance of these processes has induced me to investigate still more thoroughly the relations of the energy and its distribution between the air and the water; at first, however, as before, with the limitation to stationary waves in which the motions of the particles of water only take place parallel to a vertical plane in which the coördinates are respectively ($x$) vertical and ($y$) horizontal. Since however we can only solve even this special problem by the development into a converging series whose higher terms rapidly diminish in magnitude but offer comparatively complex forms therefore the conclusions that we may have drawn from a knowledge of the first largest term of such a series are necessarily always limited to waves of slight altitude and cause the correctness of many more important generalizations to appear doubtful.

Many of these difficulties have been surmounted in that I have been able to reduce the law of stationary rectilinear waves to a problem of minima, in which the variable quantities are the potential and actual energies of the moving fluids. From this problem in variations many general conclusions can be deduced as to the decrease and increase of the energy, and the difference between stable and unstable equilibrium of the surface of water.

Theoretically considered, there arises here a rather new problem in so far as we have to do, not with the difference between stable and un-

* From the *Sitzungsberichte* of the Royal Prussian Academy of Sciences at Berlin, 1890, vol. VII, pp. 853–872. Wiedemann, *Annalen*, 1890, XLI, pp. 641–662.

stable equilibrium of masses at rest, but with moving masses that are in steady motion.

Some examples of such differences have indeed been already treated, as in the rotation of a solid body about the axis of its greatest or least moment of inertia, and in the rotation of a fluid ellipsoid subject to gravity. But a general principle such as is given for bodies at rest, in the proposition that stable equilibrium requires a minimum of potential energy, has never yet been established for a moving system of bodies.

The following investigations lead to such propositions, which moreover can also be considered as generalizations of the propositions that I have deduced from the general equations of motion given by Lagrange in their application to the motion of "poly-cyclic" systems.*

## I. THE THEOREM OF MINIMUM ENERGY APPLIED TO STATIONARY WAVES HAVING A CONSTANT QUANTITY OF FLOW.

As in my paper of last year,† I indicate by $u$ and $v$ the component velocities of the particles of water during any motion that is free from vortices by the equations:

$$\left.\begin{aligned} u &= -\frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x}, \\ v &= \frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

I again assume, whenever the opposite is not expressly stated, that the coördinate system for $x\ y$ is at rest with reference to the wave, $x$ being vertical, positive upward, $y$ horizontal. Therefore the wave surface is at rest with reference to these coördinates while the two fluids flow steadily along it. The wave curve will be considered as periodical with the wave length $\lambda$. On the other hand, the flowing fluid will be considered as bounded by two horizontal planes whose equations are

$$x = H_1 \text{ and } x = -H_2 \quad \ldots\ldots\ldots \quad (1a)$$

Corresponding to this, I indicate the remaining quantities that refer to the fluid which is on the positive side of $x$ by the subscript 1; those that are on the negative side of $x$ by the subscript 2.

The wave-lines and these two horizontal boundary lines must be stream lines—that is to say, $\psi$ must have a constant value throughout their whole length. Since each of the functions $\psi$ can contain an arbitrary additive constant, therefore we can assume arbitrarily both of the values of $\psi$ for one of the stream lines. I assume that for the wave line for which

$$x = \bar{x}$$

we have the value

$$\psi = 0 \quad \ldots\ldots\ldots\ldots \quad (1b)$$

* *Kronecker und Weyerstrass, Journ. für Mathemat.*, 1884, vol. XCVII, p. 118.

† [See the previous paper, No. VI, in this collection of Translations.]

On the other hand, for the boundary line, for which

we have
$$\left.\begin{aligned} x &= H_1 \\ \psi_1 &= \mathfrak{p}_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1c)$$

and for the other boundary line, whose equation is

we have
$$\left.\begin{aligned} x &= -H_2 \\ \psi_2 &= \mathfrak{p}_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1d)$$

The quantities $\mathfrak{p}_1$ and $\mathfrak{p}_2$, as is well known, give respectively the volumes of the fluid that flow in the unit of time through every section between the wave surfaces for which $\psi_1 = \psi_2 = 0$, and through the upper or lower boundary surface.

These are the quantities which I have above designated as *quantities of flow*. In taking the variations of these quantities, I shall, in this paragraph, consider $\mathfrak{p}_1$ and $\mathfrak{p}_2$ as invariable.

That altitude will be adopted as the initial point for $x$, at which the boundary surface of the two quantities of fluid under consideration would be at rest, which is expressed by the equation

$$\int_{y_0}^{y_0+\lambda} x\, dy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1e)$$

that is to say, $x = 0$ is a plane such that as much water is raised above it as sinks below it.

Finally the space within which lie the quantities that are subject to variation is also bounded by two vertical planes that are separated from each other by one wave length. Since the movements are to be periodical and consistent with the wave length $\lambda$, the velocities at the right vertical surface and at the left vertical surface must be equal or

$$\frac{\partial\psi_r}{\partial x} = \frac{\partial\psi_l}{\partial x}$$

therefore for the same values of $x$

$$\psi_r = \psi_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1f)$$

and

$$\frac{\partial\psi_r}{\partial y} = \frac{\partial\psi_l}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1g)$$

According to Eq. (1) this last equation can also be written

$$\frac{\partial\varphi_r}{\partial x} = \frac{\partial\varphi_l}{\partial x}$$

or

$$\varphi_r - \varphi_l = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1h)$$

Now it is known that equations (1) are resolvable when $(\psi + \varphi i)$ can be represented as a function of $(x + yi)$, which function must show no discontinuity and no infinite values within the region filled by the fluid in question.

When the form of the wave-line is given, the values of the two functions, $\psi$, as is well known, are completely determined by the above given boundary conditions (1*b*) to (1*g*) and in that case the two integrals, which multiplied by one-half of the density of the respective fluids, give the living forces, namely:

$$\frac{2L_1}{s_1} = \int\int \left[ \left( \frac{\partial \psi_1}{\partial x} \right)^2 + \left( \frac{\partial \psi_1}{\partial y} \right)^2 \right] d\, S_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{2L_2}{s_2} = \int\int \left[ \left( \frac{\partial \psi_2}{\partial x} \right)^2 + \left( \frac{\partial \psi_2}{\partial y} \right)^2 \right] d\, S_2 \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

become absolute minima for such variations of the functions $\psi_1$, as are possible under the given circumstances, when at the same time the values $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are considered as invariable.

On the other hand the form of the wave-line is not yet determined by the conditions hitherto given, except in so far that it must be periodical with the period $\lambda$. We can however determine the form of this boundary line corresponding to the physical condition that the pressure shall be the same on either side of it, in that we require that the variation of the difference between the potential energy $\Phi$ and the living force $L = L_1 + L_2$ shall disappear

$$\delta [\Phi - L] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

The potential energy depends upon the unequal elevation of the different parts of the surface of heavier fluid above the level surface $x=0$. Its amount is easily seen to be given by the equation

$$\Phi = \tfrac{1}{2} g (s_2 - s_1) \int x^2 dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2c)$$

If $s_2$ is the denser fluid, then the positive $x$, as already remarked, must be assumed as ascending perpendicularly and $y$ must be taken as a positive quantity.

When the linear element $d\, s$ of the boundary-line of the two fluids is displaced upwards normal to its own direction by the infinitely small quantity $\delta\, N$, then the variation becomes

$$\delta\, \Phi = g (s_2 - s_1) \int x\, \delta\, N.\, d\, s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2d)$$

The variation of $L$ can be executed in two steps. In the *first* of these we imagine the boundary-line displaced in the above-given manner and first allow the two functions $\psi_1$ and $\psi_2$ in each point of space to remain unchanged, but in doing so, on that side where space is gained by the displacement $d\, s$, imagine this strip so gained to be filled with the continuous prolongation of the $\psi$ that pertains to this side, and so that the equation $\Delta\, \psi = 0$ continues to be satisfied in that region [and so that the prolongation of $\psi$ just mentioned enters here instead of the value of the other function of $\psi$ previously existing here]. This prolongation of the function $\psi$ into the strip just described is, as well known, only possible in one manner without forming discontinuities. Only when a cusp of

the appropriate function $\psi$ exists in the original boundary, therefore, especially when the boundary-line forms a sharp corner, is a continuous prolongation of the function excluded. The special physical significance of such a case we shall have to consider later on.

By this first step in the variation of $L$ we obtain

$$\delta' L=\tfrac{1}{2}\int\left[s_2\left(\frac{\partial\psi_2}{\partial N_2}\right)^2-s_1\left(\frac{\partial\psi_1}{\partial N_1}\right)^2\right] ds\,\delta N.$$

But now the values of $\psi_1$ and $\psi_2$ are no longer zero at the new boundary, but we have there, approximately

$$\psi_1=\frac{\partial\psi_1}{\partial N_1}\,\delta N$$

$$\psi_2=-\frac{\partial\psi_2}{\partial N_2}\,\delta N$$

and in order again to make these equal to zero we must execute a *second step* in the variation, such that the function $\psi$ shall so vary that these now again become zero at the new boundaries. Since according to the general laws of potential functions we have

$$\delta'' L=-s_1\int\frac{\partial\psi_1}{\partial N_1}\,\delta\psi_1\,ds-s_2\int\frac{\partial\psi_2}{\partial N_2}\,\delta\psi_2\,ds$$

therefore when we (as is necessary in our case) put

$$\delta\psi_1=-\frac{\partial\psi_1}{\partial N_1}\,\delta N$$

$$\delta\psi_2=+\frac{\partial\psi_2}{\partial N_2}\,\delta N$$

we obtain the final value:

$$\delta L=\delta' L+\delta'' L=-\tfrac{1}{2}\int\left[s_2\left(\frac{\partial\psi_2}{\partial N_2}\right)^2-s_1\left(\frac{\partial\psi_1}{\partial N_1}\right)^2\right] ds\,\delta N. \quad . \quad (2e)$$

Since finally the volume of each of the two liquids must remain unchanged during the variation, therefore it is necessary that

$$\int\delta N\,ds=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2f)$$

Hence results the variation,

$$\delta\left\{\Phi-L\right\}=-\int ds\,\delta N\left\{g\,(s_1-s_2)\,x+\frac{s_1}{2}\left(\frac{\partial\psi_1}{\partial N_1}\right)^2-\frac{s_2}{2}\left(\frac{\partial\psi_2}{\partial N_2}\right)^2+c\right\}$$

$$=-\int ds\,\delta N\,[p_2-p_1.] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2g)$$

Here $p_2$ and $p_1$ designate the fluid pressure on the upper and lower sides, respectively, of the boundary surface as they result from Euler's hydrostatic equations. Since $p_2$ and $p_1$ contain arbitrary additive constants $c$ can be omitted.

When, therefore, the equation (2*b*) is to be satisfied, that is to say, when we must have

$$\delta \left\{ \Phi - L \right\} = 0$$

then must $p_2 = p_1$ throughout the boundary surface, which is the condition of a stationary surface.

*The stability of the steady motion.*—For any form of surface that nearly corresponds to a stationary form, and which therefore still shows differences of pressure, it follows from the preceding that such a surface when it changes with the differences of the pressures experiences therefore a positive displacement $\delta N$ where $p_2 > p_1$, therefore the quantity $(\Phi - L)$ diminishes and consequently approximates to a neighboring minimum of $(\Phi - L)$, and must therefore depart from the neighboring maximum of the same quantity.

The hydro-dynamic equations show in fact that the equality of pressure in such cases can only be brought about by accelerations which act in the direction from the stronger to the feebler pressure and must disturb the steady motion.

Therefore the *stable equilibrium* of a stationary wave-form must (among all possible variations of such a form) correspond to a minimum of the quantity $(\Phi - L)$, just as in the polycyclic systems for a constant velocity of their cyclic motions. When on the other hand this same quantity $(\Phi - L)$ attains a maximum value or a cusp value for some other form of curve, then the condition of equality of pressure on both sides of the boundary surface is at least temporarily fulfilled; but individual or the very smallest disturbances of the form of equilibrium must continue to increase; the equilibrium will thus become *unstable* as is actually recognized in natural water-waves by the foaming and breaking of the crests of the waves.

On the other hand it is to be remarked that these propositions hold good only when the functions $L_1$ and $L_2$ are determined as minima in accordance with the boundary conditions of the spaces within which they hold good, and for every variation in the form of the boundary line the functions experience a change in accordance with this condition that they shall be minima.

Under the assumptions already made, the function $\Phi$ is certainly positive and finite, since only a finite quantity of liquid is present which can be raised up only through the finite altitude $H_1$. $L$ is also necessarily positive but can become $+\infty$, since the summit of the wave can approximate to the upper but the trough of the wave to the lower boundary surface and the total constant quantity of moving fluid must then be pressed with infinite velocity through infinitely narrow crevices.

The quantity $(\Phi - L)$ must therefore have a positive value for plane boundary surfaces where $\Phi = 0$, and it can become $-\infty$ for increasing wave altitudes. Whether a minimum occurs between these limits, and for what value of $\mathfrak{p}$ this could occur, can only be decided by investigation

of the individual forms of the waves. At least one cusp value occurs for a plane surface.

Only this much can be at once seen, that when an absolute minimum exists there must be a transition leading from this to the infinite negative value of $(\Phi - L)$, which transition at first begins with an ascending value and then again diminishes. There must then be a lowest value on the transition curve between the ascending and the descending values that corresponds to a maximo-minimum (absolute minimum) of the quantity $(\Phi - L)$, therefore also to a stationary form of wave, but such an one as corresponds to an unstable equilibrium, and which is on the point of becoming a breaker.

If such a minimum exists, then for it any variation in the form of the wave that makes $\Phi$ increase will make $L$ increase by the same amount. The same is true of the cusp value when we consider such waves as form trough-lines. But if we increase the values of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, that is to say, if we increase the velocity of the wind and the rate of propagation of the waves through water, then the partial differential coefficient of $L$ will be greater at both places and the two limiting values must approach each other and finally coincide, whereby the absolute minimum ceases to exist and the equilibrium becomes unstable. Hence it is to be concluded that with increasing rate of flow, stationary waves of a given wave-length will finally become impossible.

*Necessary formation of breakers when the velocity is excessive.*—That, for a constant definite value of the wave-length, minima of the function $(\Phi - L)$ are no longer possible for large values of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ exceeding a certain definite amount, can easily be shown as follows: We compute the values of $L_1$ and $L_2$ under the assumption that $\mathfrak{p}_1 = \mathfrak{p}_2 = 1$, for any arbitrarily chosen form of wave and then for an arbitrarily chosen value of $\delta\Phi$ seek the two variations of the curve which respectively make $\delta L_1$ and $\delta L_2$ to become maxima.

Among the possible variations of the form of the wave that give positive values of $\delta\Phi$ are those that give higher summits and lower troughs for the wave. Since the upper fluid has the greatest [least ?—*C. A.*] section above the summits of the waves, but the smallest [greatest ?—*C. A.*] section above their valleys therefore above the summits a greater velocity of flow must prevail than above the valleys, that is to say the value of $\frac{\partial\psi_1}{\partial N_1}$ must be greater absolutely on the summits than in the troughs. Hence follows from equation (2*e*) that when we raise the summits and depress the valleys we obtain not only positive values of $\delta\Phi$ but also positive values of $\delta L_1$ and $\delta L_2$. Consequently the desired maximum values of the two quantities $\delta L_1$ and $\delta L_2$, that belong to the prescribed positive values of $\delta\Phi$ are necessarily positive, and for a finite altitude of the wave the ratio $\frac{\delta\Phi}{\delta L_1}$ as also $\frac{\delta\Phi}{\delta L_2}$ must necessarily be finite.

We now indicate by $\alpha$ a proper fraction and imagine that we have executed a variation of $L_1$ to the amount expressed by $\alpha$, such as would correspond to the variation $\alpha.\ \delta\Phi$. On the other hand we perform the variation $\delta L_2$, to the amount $(1-\alpha)$. Then the total variation for $\Phi$ is

$$\delta\Phi = [\alpha + (1-\alpha)]\ \delta\Phi,$$

$$\delta L = \alpha.\ \delta L_1 + (1-\alpha).\ \delta L_2.$$

If now $\delta L_1 > \delta L_2$ we obtain the maximum variation of $\delta L$ when we make $\alpha = 1$; but for the opposite case we should have to make $\alpha = 0$. Thus $\delta L$ attains the greatest value that it can have for the given value of $\delta\Phi$ and the adopted form of wave.

When the greatest positive value of $\delta L$ is smaller than $\delta\Phi$ then a value for $\mathfrak{p}_1^2$ can be found that in any case will make

$$\mathfrak{p}_1^2 \delta L > \delta\Phi$$

and therefore, for at least one method of change of form, which need not necessarily be a minimal form, will make the variation $\delta\ (\Phi - L)$ negative.

Since $\Phi$ always remains finite one can always execute finite variations in its magnitude that shall be of the same order of magnitude as the displacement $\delta N$ of the elementary line $ds$, and which latter give always finite variations of $L_1$ and $L_2$, at least for finite velocities of flow along the surface.

Infinite velocities can only occur at the projecting cusps of the wave-lines and, when there is a current there, give infinite negative pressures, that is to say, the phenomena of breaking or frothing. Only when there exists no relative motion of the wave with respect to the medium into which the sharp edges of the waves project, namely, when the wind has precisely the same speed as that of the wave, can such cusp points long endure.

Except these latter cases, that lie on the boundary of breaking and frothing, we shall therefore for all continuously curved forms of waves have for every $\delta\Phi$ a maximum of $\delta L$ of the same order of magnitude. And when we seek for the smallest value of the ratio $\frac{\delta L}{\delta\varphi}$ and seek for a value of $\mathfrak{p}^2$ which shall be greater than the greatest of the values of $\frac{1}{\frac{\delta L}{\delta\Phi}}$ thus obtained, then for the corresponding strength of current the possibility of stationary wave-formation for the prescribed wave-length $\lambda$ is entirely excluded.

*Therefore stationary waves of a prescribed wave-length are only possible for such values of the velocities of flow $\mathfrak{p}_1^2$ and $\mathfrak{p}_2^2$ as are less than certain definite extreme limits.*

On the other hand, these same considerations further show that the

diminution of the values $\mathfrak{p}_1^2$ and $\mathfrak{p}_2^2$ will necessarily make the larger values of $\delta L_1$ and $\delta L_2$ with respect to $\delta\Phi$ disappear. Then variations of $\delta\Phi$ can not be counterbalanced by opposite and equal values of $L$ and then can at the most only one limiting value exist, *i. e.*, that which corresponds to the plane surface. The limit for the smallest allowable values of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ results from the preceding investigation as follows:*

$$\frac{2\pi s_1\mathfrak{p}_1^2}{H_1^2}+\frac{2\pi s_2\mathfrak{p}_2^2}{H_2^2}=g\cdot\lambda(s_1-s_2).$$

*Hereby the range of values of* $(\mathfrak{p}_1)^2$ *and* $(\mathfrak{p}_2)^2$ *that permit stationary waves of the wave-length* $\lambda$ *is limited on its lower side.*

It is to be noticed that the quantity $\mathfrak{p}_2$ determines the progressive velocity of the wave with respect to the water; $\mathfrak{p}_1$, on the other hand, determines the velocity of the wind relative to the wave. Either of these can be small if the other is sufficiently large.

## II. THE THEOREM OF MINIMUM ENERGY APPLIED TO STATIONARY BILLOWS WITH A CONSTANT VALUE OF THE VELOCITY POTENTIAL.

The value of the living force, as given in equation (2), can by partial integration be written

$$L_1=\frac{s_2}{2}\int\mathfrak{p}_1\cdot\frac{\partial\psi_1}{\partial x}\cdot dy,$$

in which the integral relates only to the upper horizontal boundary line. The portions of the integral for the other limit of the space $S_1$ all disappear. Since now according to equation (1)

$$\frac{\partial\psi}{\partial x}=\frac{\partial\varphi}{\partial y}$$

there results

$$L_1=\frac{s_1}{2}\mathfrak{p}_1\int\frac{\partial\varphi_1}{\partial y}\cdot dy.$$

Of, if we put

$$\mathfrak{f}=\varphi_{y+\lambda}-\varphi_y,$$

which difference is independent of $x$, we obtain

$$L_1=\frac{s_1}{2}\mathfrak{p}_1\cdot\mathfrak{f}_1 \quad\ldots\ldots\ldots\ldots\ldots \quad (3)$$

and similarly

$$L_2=-\frac{s_2}{2}\mathfrak{p}_2\mathfrak{f}_2 \quad\ldots\ldots\ldots\ldots\ldots \quad (3a)$$

* My attention has been called to the fact that Sir William Thomson has already given this equation as the first approximation, taking into consideration the strength of the wind, *Philos. Mag.*, 1871 (4), vol. XL, p. 362, where, moreover, the influence of capillarity is also considered.

The quantities $\mathfrak{p}$ and $\mathfrak{f}$ are dependent on each other as soon as the form of the space is given for whose boundary they hold good; so that we can put

$$\mathfrak{p}=\mathfrak{f}.\ \mathfrak{R}$$

where $\mathfrak{R}$ indicates a value that depends only on the size and form of this space. Hence there results

$$L=\frac{s}{2}\mathfrak{p}.\ \mathfrak{f}=\frac{s}{2}\mathfrak{f}^2\,\mathfrak{R}=\frac{s}{2}\frac{\mathfrak{p}^2}{\mathfrak{R}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3b)$$

When therefore $\mathfrak{R}$ experiences a change $\delta\ \mathfrak{R}$ then if $\mathfrak{f}$ remains unchanged we have

$$\delta\, L=\frac{s}{2}.\ \mathfrak{f}^2.\ \delta\,\mathfrak{R}$$

$$\delta\ \mathfrak{f}=0;$$

on the other hand, when $\mathfrak{p}$ remains unchanged we have

$$\delta\ L=\frac{s}{2}\ \frac{\mathfrak{p}^2\ .\ \delta\,\mathfrak{R}}{\mathfrak{R}^2}=-\frac{s}{2}\ \mathfrak{f}^2.\ \delta\,\mathfrak{R}$$

$$\delta\,\mathfrak{p}=0.$$

Both variations therefore have the same values with opposite signs. We can therefore, instead of

$$\delta\Phi-\delta\ L=0$$

$$\delta\,\mathfrak{p}_1=\delta\ \mathfrak{p}_2=0.$$

which is the form of variation for the stationary condition where the variation of $\delta\ L$ is deduced from the variation of the form of the region, also write

$$\delta\ \Phi+\delta\ L=0$$

$$\delta\ \mathfrak{f}_1=\delta\ \mathfrak{f}_2=0.$$

The quantities $\mathfrak{f}$ according to their definition have the value:

$$\mathfrak{f}=\int_{x,\,y.}^{x,\ y\,+\,\lambda}(u.\ dx+v.\ dy)$$

the integral being taken for any value that leads from the point $(x, y)$ to the point $(x, y+\lambda)$. When we choose the stream-line $\psi=$ constant for this path between these points then the integral also indicates a path along which a series of material liquid particles would flow. The value of the integral $\mathfrak{f}_1$, as computed for such a series of material flowing particles as is well known remains unchanged, whatever motions may otherwise be going on in the liquid, provided there are no differences in the sum total of the pressures and potentials of the exterior forces between the beginning and the end of the series, and provided there is no friction. This is the same sum that also remains unchanged in the vortex motion in every closed ring of material particles. We can therefore in fluid motions consider $s_1\ \mathfrak{f}_1$ and $s_2\ \mathfrak{f}_2$ as the moments of

motion, which remains invariable except for the influence of direct accelerating forces, while the quantities of flow $\mathfrak{p}_1$ and $\mathfrak{p}_2$ thereby receive the significance of velocities. Thus the two problems in variations, here solved, are completely analogous to the propositions developed by me in the theory of polycyclic systems, that

$$\delta(\Phi - L) = -\Sigma [P_a \delta p_a] \qquad \text{(3e)}$$
$$\delta q_a = 0$$

when the velocities $q_a$ of the cyclic motions are maintained constant. In this equation $p_a$, are the variable coördinates, and $P_a$ are the forces tending to increase these coördinates. Stable equilibrium, as is easy to see, corresponds to a minimum of the $(\Phi - L)$.

On the other hand, when we assume the moment of motion $\frac{\partial L}{\partial q_a}$ to be constant we have

$$\delta(\Phi + L) = -\Sigma [P_a d p_a]$$

$$\delta\left(\frac{\partial L}{\partial q_a}\right) = 0 \qquad \text{(3f.)}$$

Here, also, stable equilibrium demands that the quantity $(\Phi + L)$, that is to say, the total energy of the body be a minimum.

The equation $(2g)$ corresponds throughout to the above equation $(3e)$ for polycyclic systems, only that in the former the number of variable coördinates $\delta N$ of the surface elements $ds$ is infinitely large and the force which in it corresponds to $P_a$, namely, the fluid pressure, is a continuous function of $y$; hence the integral is used instead of the sign of summation.

That stable equilibrium, even in the theory of waves, also corresponds to the minimum of energy for a constant value of $\mathfrak{f}$ is established when we think of the influence of friction which can restore a disturbed stable equilibrium but not a disturbed unstable equilibrium. Friction always diminishes the store of energy that may be present. It can, therefore, restore a disturbed minimum of energy but not a departure from a maximum.

### III. THE THEOREM OF MINIMUM ENERGY APPLIED TO LAYERS OF INFINITE THICKNESS.

In the following we shall consider the two layers of fluid on whose boundary surface the waves form, as very deep in the vertical direction, therefore the values $H_1$ and $H_2$ as very large and as respectively increasing beyond all limits to infinity, in order to free the theory of waves from those complications which are brought about by the influence of the upper and lower horizontal boundary surfaces.

Under these circumstances the motion on these two far distant horizontal boundary surfaces does not differ sensibly from rectilinear uni-

form velocity. For the surface $H_1$ we put $a_1$ for this velocity, for the surface $H_2$ we take $(-a_2)$ since we give the latter a motion in the opposite direction to that which would be given to it in the normal cases where the wind outruns the wave.

We have at once

$$+\mathfrak{f}_1 = a_1 . \lambda$$
$$-\mathfrak{f}_2 = a_2 . \lambda$$

and in the higher layers of the fluid

$$\psi_1 + \varphi_1 \, i = + a_1 \, (x + yi) + h_1$$

where $h_1$ is a constant to be determined by the equation $(1e)$.

Similarly

$$\psi_2 + \varphi_2 \, i = - a_2 \, (x + yi) + h_2$$

For plane boundary surfaces when for these as above assumed $\psi_1 = \psi_2 = 0$, and also $x = 0$, we should also have $h_1$ and $h_2$ both equal to zero, and the living force in this case becomes

$$L_1{}^1 = \frac{s_1}{2\mathfrak{p}_1} \cdot \mathfrak{f}_1 = \frac{s_1}{2} \, a_1^2 . \, H_1 \, \lambda$$

$$L_2{}^1 = -\frac{s_2}{2} \mathfrak{p}_2 . \, \mathfrak{f}_2 = \frac{s_2}{2} \, a_2^2 . \, H_2 \, \lambda$$

When on the other hand billows have arisen, $L_1$ is smaller for a constant value of $a_1$ and therefore also of $\mathfrak{f}_1$, since, as we have seen then a negative value of $\delta L_1$ results from an increase in the altitude of the wave. We can therefore under these circumstances put

$$L_1 = \frac{s_1}{2} a_1^2 \, (H_1 - r_1) . \, \lambda$$

wherein $r_1$ has a positive value that depends on the form and height of the wave, but not on $H_1$. If we imagine $H_1$ increased by the quantity $D\,H_1$ and the quantity $L_1$ correspondingly increased by $D\,L_1$ then in the strip thus added to the field the velocity is uniformly equal to $a_1$ and therefore

$$D\,L_1 = \frac{s_1}{2} a_1^2 . \, D\,H_1$$

$$L_1 + D\,L_1 = \frac{s_1}{2} . a_1^2 \left[ (H_1 + D H_1) - r_1 \right] . \lambda .$$

Therefore the same value of $r_1$ also holds good for the greater altitude independent of the value of $D\,H_1$.

The formula (4) gives directly

$$\mathfrak{p}_1 = -\mathfrak{f}_1 \, (H_1 - r_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Compared with galvanic conditions, $\mathfrak{p}_1$ measures the total flow or the intensity of the current; $\mathfrak{f}_1$ is the difference of potential between the boundary surfaces. Hence $(H_1 - r_1)$ is the conductivity which is pro-

portional to the sectional area. Therefore $r_1$ corresponds to that constant diminution of the sectional area which causes the current to diminish just as the irregular obstruction by the waves does.

For a constant value of $a_1$ and $a_2$, respectively, since $\lambda$, $H_1$, and $H_2$ remain unchanged, the condition that a minimum of $(\Phi + L)$ should exist gives

$$\delta(\Phi+L)=\delta\Phi-\frac{s_1}{2}a_1^2\delta r_1-\frac{s_2}{2}a_2^2\delta r_2=0 \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

The other minimum condition in which the $a$ are to be replaced by

$$a=\frac{\mathfrak{p}}{H-r}$$

is

$$\delta(\Phi-L)=\delta\Phi-\frac{s_1}{2}\mathfrak{p}_1^2\frac{\delta r_1}{(H_1-r_1)^2}-\frac{s_2}{2}\mathfrak{p}_2^2\frac{\delta r_2}{(H_2-r_2)^2}$$

which agrees perfectly with that first found.

The quantities $r_1$ and $r_2$ depend only on the form of the wave, and are generally found by simple computations as soon as we have found the form of the functions $\psi_1$ and $\psi_2$.

*Horizontal transportation of the superficial layer.*—The quantity of flow $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the two fluids is no longer the same as it would be over plane surfaces of water for equal values of the velocities $a_1$ and $a_2$, but it is smaller than before in the upper medium by the quantity $r_1\,a_1$ and in the lower medium by the quantity $r_2\,a_2$.

Imagine now the velocity $(-a_2)$ added to both sides so that the lower medium comes to rest, but the waves progress with the velocity $(-a_2)$. Then beneath plane boundary surfaces all motion disappears, but beneath billowy surfaces a general current is set up of the magnitude $-a_2\,r_2$, and thus the wind in the upper region travels not with a uniform velocity $(a_1+a_2)$, but just above the billowy surface there occurs a diminution of the flow of air to the amount of $a_1\,r_1$.

These two currents cause the mass of air and water taken together to have a different moment of motion in a horizontal direction than if they flowed with the same velocities $a_1$ and $a_2$ over plane boundary surfaces, and this difference of moment of motion, reckoned as positive in the direction of the wind, is

$$M = s_2\,a_2\,r_2 - s_1\,a_1\,r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This can only be equal to zero when

$$s_2\,a_2\,r_2 = s_1\,a_1\,r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a),$$

or, if we introduce $w$, the velocity of the wind,

$$w = a_1 + a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b),$$

then equation (5$a$) becomes

$$\frac{s_2\, r_2\, w}{s_1\, r_1 + s_2\, r_2} = a_1$$

$$\frac{s_1\, r_1\, w}{s_1\, r_1 + s_2\, r_2} = a_2.$$

Since now $r_1$ and $r_2$ have values that differ but little for the ordinary waves (as the subsequent computations will show), and since for air and water

$$\frac{s_1}{s_2} = \frac{1}{773.4},$$

therefore this condition gives the rate of propagation of the wave against the water as approximately

$$a_2 = \frac{w}{774.4}.$$

For waves of low altitude equation I, Section VII of my paper of the previous year,* neglecting the small quantities $z$ and $\rho$, becomes

$$s_1\, a_1^2 + s_2\, a_2^2 = \frac{g.\lambda\, (s_2 - s_1)}{2\pi}$$

If we put $w$=10 metres which corresponds to a rather strong wind, then for low waves of a constant moment of motion, we have

$$a_1 = 9^{m}.98709$$
$$a_2 = 0^{m}.01291$$
$$\lambda = 0^{m}.082782$$

These waves of only 8 centimeters in length evidently can correspond only to the first crumpling of the surface, such as a strong wind striking upon it immediately excites. Only when the same wind blows for a long time over these initial waves, and gives them a part of the moment of motion of a long stretch of air, can waves be thereby produced with greater velocities of propagation.

Hence in accordance with experience it follows that wind of a uniform strength striking a quiet surface of water can only produce more rapidly running waves, namely, those that are longer and higher, when it has acted for a long time on the waves that first arose, and has accompanied these for a long distance over the surface of the water.

At the same time it also becomes clear that for a uniform wind the waves can only increase in size when the wind advances faster in the same direction than the waves themselves.

*Energy of progressive waves on quiet water.*—As in the case of the moment of motion, so also with the storage of energy in the wave. Our previous comparisons of the energy of different waves among themselves has reference to the energy of relative motion of the fluid with reference to the stationary wave.

* [See page 107 of this collection of Translations.]

The well known proposition that *the living force of any complex mechanical system is equal to the living force of the motions relative to its center of gravity plus the living force of the motion of the center of gravity at which we imagine the whole mass of the system to be concentrated*, can, with only a small change in the method of expression, be applied to our case. For since the total mass of the system multiplied by the velocity $\mathfrak{v}$ of the center of gravity, gives the amount of the total momentum of the system in the direction of this velocity, therefore we can also put the living force $\mathfrak{L}$ of the center of gravity

$$\mathfrak{L} = \tfrac{1}{2} M \mathfrak{v} = \tfrac{1}{2} \mathfrak{M} \mathfrak{v}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $M$ is the momentum of the whole system in the direction of $\mathfrak{v}$ and $\mathfrak{M}$ is the mass of the system. If we now compare with each other two different conditions of motion and configuration of the system in which $L_1$ and $L_2$ are the living forces of the motions relative to the center of gravity, $\Phi_1$ and $\Phi_2$ are the potential energies, $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are the parallel velocities of the center of gravity, then the difference in the total energy of the system in the two conditions is

$$E_1 - E_2 = \Phi_1 - \Phi_2 + L_1 - L_2 + \tfrac{1}{2} \mathfrak{M} . \mathfrak{v}_1^2 - \tfrac{1}{2} \mathfrak{M} . \mathfrak{v}_2^2 .$$

If now, without changing the relative motions, I in both cases add the quantity $c$ to the velocity of the center of gravity, then the above difference of energies changes into

$$E_1' - E_2' = E_1 - E_2 + c\,(M_1 - M_2).$$

If $(M_1 - M_2) = 0$, then the value of the difference in energy is not changed by the addition of the velocity $c$. This must be true even when $H_1$ and $H_2$, and therefore the masses of the moving fluids, increase to infinity, since for our undulating fluids the differences $(E_1 - E_2)$ and $(M_1 - M_2)$ are finite for each wave length.

Therefore the difference of the energy for stationary waves and for stationary deep water will be equally great only for waves that satisfy the condition (5*a*). According to the propositions above deduced, stationary waves of this kind must have less energy than smooth water, which is therefore also true in this case for this kind of waves above quiet water.

For waves that have larger values of $a_2$, the addition of a common velocity $(-a_2)$, which brings the deep water into rest, changes the difference of energy between the two states, that of a smooth surface and that of a wave formation, by the quantity

$$E_1' - E_2' = E_1 - E_2 + a_2\,[s_2 a_2 r_2 - s_1 a_1 r_1].$$

The index 1 refers to the billowy surface, the index 2 to the plane surface, the accented $E'$ refers to quiet deep water, the non-accented $E$ refers to stationary waves.

Hence it results that when waves of considerable progressive velocity trench upon quiet deep water the generally very small differences $(E_1-E_2)$ lose their negative and assume a positive value.

Here also the energy that is given to the previously quiet water in the form of an elevation of its surface and the living force of its motion must be abstracted from the atmosphere. In order to obtain a sufficient amount for the formation of large waves, it will on this account be necessary that long layers of air shall blow over and shall give up a part of their living force.

In the first moment when a new gust strikes the surface of the water stationary waves only can be formed for which $M=0$ and $E_1-E_2=0$ and $a_2$ has the value given in equation ($5a$). The last condition shows that these waves will be near the point of spirting, as we in fact often see in the case of small ripples suddenly excited on the surface of the water. Moreover in these small ripples, as Sir William Thomson has shown, the capillary tension of the liquid comes into consideration, which somewhat increases the store of energy of the billowy surface.

In general therefore, stationary waves are not formed immediately at the beginning, since the waves of constant momentum would leave behind an excess of energy. But when from the very beginning waves that have partly a positive and partly a negative difference of momentum and of energy are successively produced on the quiet water, then the sum of these differences can become zero. These systems of waves, having different wave-lengths and progressive velocities, cause manifold interferences as they progress, and, according to the principle given by me for combination-tones (which in its application to the tidal wave has already received a very beautiful confirmation by Sir William Thomson's analysis of the tidal observations collected by the British Association), waves of greater wave-length can gradually be formed.

So long as the wind outruns the waves it steadily increases the store of energy and the momentum of the waves, and furthermore, so long as the energies computed for stationary waves diminish and can form a still lower minimum, the inclination to attain the form of least energy under the coöperation of all the small perturbations which the other concurrent waves bring about, in the case of nature, will develop still further. This will finally lead to the value corresponding to the formation of a cusp and to the foaming of the upper ridge in case this can be produced by the given wind velocity.

In April of this year [1890] I endeavored by observations that I instituted at the Cape of Antibes [near Marseilles] to arrive at some conclusion as to these consequences drawn from theory. With a small portable anemometer I measured the strength of the wind directly at the edge of the steep cliff of the narrow tongue of land which projects rather far into the sea. However, the observations showed that many times a stronger wind must have prevailed out on the sea than I had been able to observe on shore. I also counted the number of approaching billows.

With water-waves the same as with sound-waves it is to be assumed that, through all deviations, delays, and diminutions that they experience, the time of vibration remains unchanged. This time may therefore be determined near the shore even though the progressive velocity in shallow water is changed and the form and the length of the waves change. The number, $N$, of the waves in a minute is expressed by

$$N = \frac{60.a_2}{\lambda}$$

When $a_2$ increases to $na_2$ then $\lambda$ increases to $n^2\lambda$, as shown in my paper of a year ago, and therefore

$$N_n = \frac{N}{n}.$$

A velocity $a_2=10$ metres would give 9.4 waves per minute; on the other hand a velocity $a_2=5$ would give 18.8.

The counting of the waves without registering instruments is now not to be executed with great accuracy, since on the sea, so far as I have seen it, there are always numerous adjacent waves of rather different periodic times which interfere and give phenomena corresponding to the acoustic beats. During the minimum of motion one can easily make errors in the counting; by repeated countings at the same place we obtain therefore variations of about one-tenth or even more of the desired number.

The strength of the wind that I observed on the shore did not exceed 6.1 metres per second. This was on the evening of my arrival in Antibes, April 1, 1890; the wind was from east southeast; I counted between 8.5 and 10 waves per minute. On the next morning, April 2, there were still 10 to 10.5 waves per minute, although the wind had almost entirely gone down. This number of waves would be explicable only when a wind about 10 metres per second had blown steadily over the open sea. On the 2d of April the wind rose in the course of the day to a velocity of only 4 metres per second. Yet on the 3d of April also the number of waves was still 9.5 with a very feeble wind; on the 4th of April for the first time an increase was perceptible up to 12.3 waves per minute.

During a series of quiet days the number of steadily diminishing waves gradually increased to 17 or 18. Finally on the 7th of April the wind began again to increase. In the morning I found a velocity of 3.3 metres per second, which in the course of the day increased to 5.5 and brought the number of waves down to 11 5. This time, however, the location of the increased wind was demonstrable. In Marseilles during the previous night a severe whirlwind had prevailed and the larger waves excited by it stretched as a sharply defined dark-gray band from the sea horizon hitherward and reached Cape Antibes about midday, long before the stronger wind that had given rise to them and which had morever at the latter place by no means the same force as in Marseilles.

These few observations therefore show a connection between the number of waves per minute and the strength of the wind and even an agreement, at least in the order of magnitude. But the numbers of waves are all somewhat smaller than they should be as computed from the strength of the wind on shore and leave us to conclude that a stronger wind must have prevailed in the open sea. They show however also that the re-action of a strong wind may last many days.

For a progressive velocity of 10 metres the waves would in one day travel $7\frac{3}{4}$ degrees of longitude. Therefore, had the Mediterranean even to the Gulf of Sidra been on the 1st of April covered with waves excited by a strong breeze of 10 metres velocity, these would need two and a half days before the last ones would reach the coast of southern France.

It will of course be possible to solve the problem more thoroughly only when we have at hand continuous registers of the billows and extended observations of the velocity of the wind. These latter are unfortunately not yet collected for the month of April of this year, or at least not yet published, and could therefore not be used by me.

VIII.

# THE THEORY OF FREE LIQUID JETS.*

By Prof. G. KIRCHHOFF.

Helmholtz in his communication on discontinuous motions in liquids, *Berlin, Monats-berichte*, April, 1868,† has for the first time determined the form of a free jet of liquid in a special case. The method used by him in this determination can, as will here be shown, be so generalized that it leads to the solution of the same problem for a large number of cases.

It is assumed that the fluid is incompressible, that no exterior forces act upon it, that its particles do not rotate, that the currents are steady, and finally, that the movement is everywhere parallel to a fixed plane.

Let $x$ and $y$ be the rectangular coördinates of any point of the space occupied by the flowing liquid reckoned parallel to the fixed plane and let $\varphi$ be the velocity potential at this point, then $\varphi$ is a function of $x$ and $y$ such that it satisfies the equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

In this equation $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ are the velocities parallel to the axes of $x$ and $y$ and if $p$ is the pressure and $\rho$ is the density, then we have further

$$p=c-\frac{\rho}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right],$$

where $c$ indicates a constant. If the flowing liquid has a free boundary then this must correspond to a stream line and the pressure must be constant throughout it. The second of these conditions, if we adopt a proper system of units, will be expressed by the equation

$$\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=1.$$

* From Borchardt's *Journal*, 1869, vol. LXX, or Kirchhoff *Gesammelte Abhandlungen*, Leipsic, 1882, pp. 416–427.

† [See also No. III of this present collection of Translations.]

The partial differential equation for $\varphi$ is satisfied if we have

$$z=x+iy \qquad \omega=\varphi+i\psi,$$

where $i=\sqrt{-1}$, and $\omega$ can be any function of $z$. Therefore the equation of any curve of flow or stream line is $\psi=$constant, and we have

$$\frac{\partial\varphi}{\partial x}=\frac{\dfrac{\partial x}{\partial\varphi}}{\left(\dfrac{\partial x}{\partial\varphi}\right)^2+\left(\dfrac{\partial y}{\partial\varphi}\right)^2}$$

$$\frac{\partial\varphi}{\partial y}=\frac{\dfrac{\partial y}{\partial\varphi}}{\left(\dfrac{\partial x}{\partial\varphi}\right)^2+\left(\dfrac{\partial y}{\partial\varphi}\right)^2}$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=\frac{1}{\left(\dfrac{\partial x}{\partial\varphi}\right)^2+\left(\dfrac{\partial y}{\partial\varphi}\right)^2}$$

if we assume that $x$ and $y$ on the right-hand side of these equations can be represented as functions of $\varphi$ and $\psi$. Therefore the conditions for a free boundary of the jet are that for it $\psi=$constant, and

$$\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2=1.$$

The problem is therefore to express $\omega$ as such a function of $z$ as will satisfy these conditions.

To this end we put

$$\frac{dz}{d\omega}=f(\omega)+\sqrt{f(\omega)f(\omega)-1}$$

and select the function $f(\omega)$ so that it is real for a certain value of $\psi$ and for a certain range of $\varphi$, and so that it lies between the limits $-1$ and $+1$. For this value of $\psi$ and for this range of $\varphi$ we have

$$\frac{\partial x}{\partial\varphi}=f(\omega), \qquad \frac{\partial y}{\partial\varphi}=\sqrt{1-f(\omega)f(\omega)}$$

whence

$$\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2=1;$$

that is to say, the stream line corresponding to the value of $\psi$ can form a free boundary to the moving liquid in that portion which corresponds to the range of $\varphi$. If there are many values of $\psi$ for which $f(\omega)$ has the described property then all the stream lines that correspond to these values can be free boundaries.

In general $\omega$ is defined by the equation above given for

$$\frac{dz}{d\omega}$$

as a many-valued function of $z$ for any definite assumption as to $f(\omega)$. Let the region of $z$, that is to say the space filled with the moving liquid,

be so bounded that, within it, no branch of $\omega$ merges into another; such a branch, therefore, represents a possible mode of fluid motion. The desired object will be attained when the region of $\omega$ is appropriately bounded.

In reference to the boundary of the region of $\omega$ it is recognized, first, that it is a line that returns into itself and without cutting itself and that consists of parts for which $\psi$ has a constant value and of parts for which $\varphi$ has an indefinitely large positive or an indefinitely large negative value.

Within the region of $\omega$, $f(\omega)$ is a single-valued function of $\omega$. If we had adopted an expression for $f(\omega)$ that represented a many-valued function, then at its cusp point should start the sections for which $\psi$ has a constant value.

Furthermore $\sqrt{f(\omega)f(\omega)-1}$ should also be made a single-valued function of $\omega$, in that through those points for which $f(\omega)=\pm 1$, the sections pass for which $\psi$ has a constant value. For any point of the region of $\omega$ the sign of the radical quantity is still at our disposal. If points occur for which $f(\omega)$ is infinite or infinitely great,* then for one of these points we may make

$$\sqrt{f(\omega)f(\omega)-1}=+f(\omega)$$

and assume that this equation holds good for them all.

It is further assumed that the function $f(\omega)$ is only infinite at its cusp points if it is so anywhere, and even here it is infinite only in such a way that if $f(\omega_0)$ is infinite then $(\omega-\omega_0)f(\omega)$ approximates to zero when $\omega$ has a value approximating that of $\omega_0$.

Within the designated region of $\omega$ therefore $z$ is a single-valued function of this variable and such that it is never infinite.

Now consider $\omega$ as a function of $z$. The region of $z$ that corresponds to the adopted region of $\omega$ does not extend through infinity, and is bounded by a line that returns into itself and which is made up of the lines whose equations are $\varphi=-\infty$ and $\varphi=+\infty$ and of stream lines; a certain portion of the latter can be considered as a free boundary of the moving fluid, the other part can be considered as a fixed wall. Within this region of $z$, $\omega$ has no cusp point, since at no point of it does $\frac{dz}{d\omega}$ become zero. Therefore under the condition that the boundary of the region of $z$ shall not intersect itself, $\omega$ becomes within that region a single valued function of $z$.

This function of $z$ is completely determined as soon as one has found a single value of $z$ corresponding to a given value of $\omega$.

(I.) An example that constitutes a generalization of the case treated of by Helmholtz is obtained if we put

$$f(\omega)=k+e^{-\omega}$$

* By infinite, I designate the reciprocal of zero, but by infinitely great, the reciprocal of an infinitely small quantity.

where, as also in the following examples, $k$ indicates a positive real fraction, and where the region of $\omega$ is bounded by the lines

$$\psi=0 \quad \varphi=-\infty$$
$$\psi=\pi \quad \varphi=+\infty$$

The expression adopted for $f(\omega)$ is single valued. The multiple points of $\sqrt{f(\omega)f(\omega)-1}$ that do not lie outside the region of $\omega$ are the points

$$\varphi=-\log(1-k) \quad \psi=0$$
$$\varphi=-\log(1+k) \quad \psi=\pi$$

These lie in the boundary of this region, and, therefore, it need not be further bounded by sections.

The equations of the boundary of the region of $\omega$ are also the equations of the boundary of the region of $z$. If we assume that for $\varphi=-\log(1+k)$ and $\psi=\pi$ we have $x=0$ and $y=0$, then these equations when developed become the following

For $\psi=\pi$ and $\varphi<-\log(1+k)$ there results

$$y=0 \text{ and } x=\int_{-\log(1+k}^{\phi}(k-e^{-\phi}-\sqrt{(k-e^{-\phi})^2-1})\,d\varphi$$

where the root (as also hereafter every root of a positive quantity), is taken to be positive. By these equations the positive half of the axis of $x$ is represented; this is to be taken as a fixed wall; at the initial point of coördinates it merges into the free boundary. For this free boundary, namely, for $\psi=\pi$ and $\varphi>-\log(1+k)$ we have

$$x=\int_{-\log(1+k)}^{\phi}(k-e^{-\phi})\,d\varphi$$

$$y=-\int_{-\log(1+k)}^{\phi}\sqrt{1-(k-e^{-\phi})^2}\,d\varphi$$

Furthermore for $\psi=0$ and $\varphi<-\log(1-k)$ we have

$$x=\int_{-\log(1-k)}^{\phi}\left(k+e^{-\phi}+\sqrt{(k+e^{-\phi})^2-1}\right)d\varphi+a$$

$$y=b$$

and for $\psi=0$ and $\varphi>-\log(1-k)$

$$x=\int_{-\log(1-k)}^{\phi}(k+e^{-\phi})\,d\varphi+a$$

$$y=-\int_{-\log(1-k)}^{\phi}\sqrt{1-(k+e^{-\phi})^2}\,d\varphi+b$$

where

$$a=k\log\frac{1+k}{1-k}-2-\pi\sqrt{1-k^2}$$

$$b=-2\pi k$$

The first part of the stream line $\psi=0$ which is a straight line parallel to the axis of $x$ and extending to the point $x=a$, $y=b$, is to be considered as a fixed wall; the second part is to be considered as the free boundary of the outflowing jet.

The approximate course of the lines $\psi=\pi$ and $\psi=0$ is shown in Fig. 4.

The completion of the boundary of the region of $z$ is formed by the line $\varphi=-\infty$, namely,

$$x=2\,k\,\varphi-2e^{-\phi}\cos\psi+a_1$$

$$y=2\,k\,\psi+2e^{-\phi}\sin\psi+b_1$$

and by the line, $\varphi=+\infty$, namely,

$$x=k\,\varphi+\sqrt{1-k^2}.\psi+a_2$$

$$y=k\,\varphi-\sqrt{1-k^2}.\varphi+b_2$$

where $a_1$, $b_1$, $a_2$, $b_2$ are constants whose values are easily obtainable and which are partly used in the computation of $a$ and $b$. The first of these two lines can be defined as a half circle that is described with an infinitely large radius about the origin of coördinates; the second is a straight line that is perpendicular to the jet at an infinitely great distance from the origin; at this distance the jet forms an angle with the positive axis of $x$ whose cosine equals $k$.

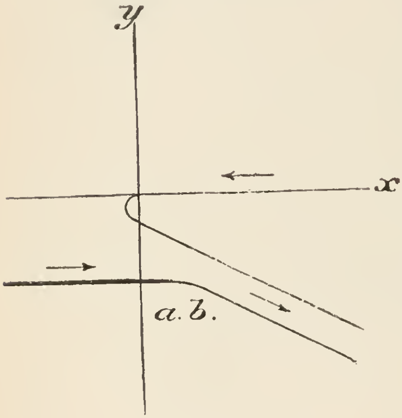



Fig. 4.

If we assume that $k$ equals 1 then $a$ becomes infinite and the point $(a, b)$ removes to infinity; the region of $\omega$ can in this case be bounded by the lines $\psi=\pi$ and $\psi=-\pi$ instead of by the lines $\psi=\pi$ and $\psi=0$; thus we come to the case treated of by Helmholtz and illustrated by Fig. 5.

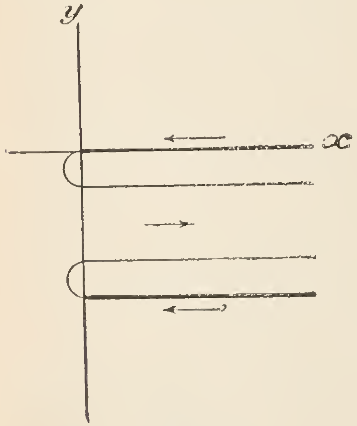



Fig. 5.

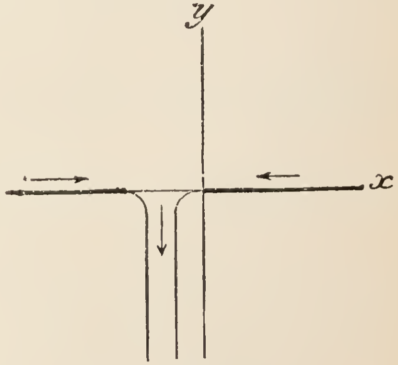



Fig. 6.

If we make $k$ equal zero then will $b$ equal zero; in this case the boundary of the moving fluid is represented by Fig. 6.

(II.) As a second example the case where

$$f(\omega) = k + \frac{1}{\sqrt{w}}$$

will be treated and the region of $\omega$ stretches indefinitely far in all directions.

In order to make $f(\omega)$ a single-valued function we draw a section from the point $\omega=0$, for which section $\psi=0$ and $\varphi>0$ and assume that for $\varphi=+0$ and $\psi=+0$ the real part of $\sqrt{\omega}$ is positive. The cusp points of the curve $\sqrt{f(\omega)f(\omega)-1}$ are the points for which $\omega=0$, $\frac{1}{\sqrt{w}}=1-k$, $\frac{1}{\sqrt{w}}=-(1+k)$; therefore they all lie on the section already drawn therefore do not require the making of a new section. As concerns the sign of $\sqrt{f(\omega)f(\omega)-1}$ it must be so determined according to the adopted rules that the real part of this radical quantity shall be positive for $\varphi=+0$, and $\psi=+0$. Finally it is assumed that $\omega$ and $z$ disappear simultaneously.

The line for which $\psi=0$, and $\varphi>0$, is the boundary of the region of $z$. This line is composed of many parts which are to be distinguished from each other. For $\psi=+0$, and $0<\varphi<\frac{1}{(1-k)^2}$ we have

$$x=\int_0^\phi \left(k+\frac{1}{\sqrt{\varphi}}-\sqrt{\left(k+\frac{1}{\sqrt{\varphi}}\right)^2-1}\right)d\varphi$$

$$y=0,$$

Then again for $\psi=-0$, and $0<\varphi<\frac{1}{(1-k)^2}$

$$x=\int_0^\phi \left(k-\frac{1}{\sqrt{\varphi}}-\sqrt{\left(k-\frac{1}{\sqrt{\varphi}}\right)^2-1}\right)d\varphi$$

$$y=0.$$

These equations represent a part of the axis of $x$ which is to be adopted as the fixed wall. If we use the relation

$$\int\sqrt{\left(k+\frac{1}{\sqrt{\varphi}}\right)^2-d\varphi}=$$

$$\frac{(1-k^2)\sqrt{\varphi}-k}{1-k^2}\sqrt{(k\sqrt{\varphi}+1)^2-\varphi}+\frac{1}{(1-k^2)^{\frac{3}{2}}}\text{ arc sin}\left((1-c^2)\sqrt{\varphi}-k\right)$$

we find for the end of this part (of the axis of $x$) the expression

$$x=2\frac{1+k-k^2}{(1-k)(1-k^2)}+\frac{1}{(1-k^2)^{\frac{3}{2}}}\left(\frac{\pi}{2}+\text{ arc sin } k\right)$$

and

$$x=-2\frac{1-k-k^2}{(1+k)(1-k^2)}-\frac{1}{(1-k^2)^{\frac{3}{2}}}\left(\frac{\pi}{2}-\text{ arc sin } k\right)$$

where the arc whose sine is $k$ is to be taken between zero and $\frac{\pi}{2}$.

For $\psi=+0$ and $\varphi>\frac{1}{(1-k)^2}$ we have

$$\frac{dx}{d\varphi}=k+\frac{1}{\sqrt{\varphi}};\ \frac{dy}{d\varphi}=-\sqrt{1-\left(k+\frac{1}{\sqrt{\varphi}}\right)^2}$$

and for

$$\psi=-0 \text{ and } \varphi>\frac{1}{(1+k)^2} \text{ we have}$$

$$\frac{dx}{d\varphi}=k-\frac{1}{\sqrt{\varphi}};\ \frac{dy}{d\varphi}=-\sqrt{1-\left(k-\frac{1}{\sqrt{\varphi}}\right)^2}$$

The lines that are represented by the integrals of these equations, when we determine the constants of integration so that these lines start from the previously indicated termini of the fixed walls, are the free boundaries of the moving liquid. The other boundaries of the region of $z$ lie at infinite distances, as is seen from the fact that when $\omega=\infty$ we have

$$\frac{dz}{d\omega}=k-i\sqrt{1-k^2}$$

this equation shows at once that at an infinitely great distance from the origin of coördinates the flow takes place with the velocity 1 in a direction that forms an angle with the axis of $x$ whose cosine is $k$. Figure 7 illustrates the boundary of the region of $z$; besides this boundary the figure also gives the stream line for which $\psi=0$, and $\varphi<0$.

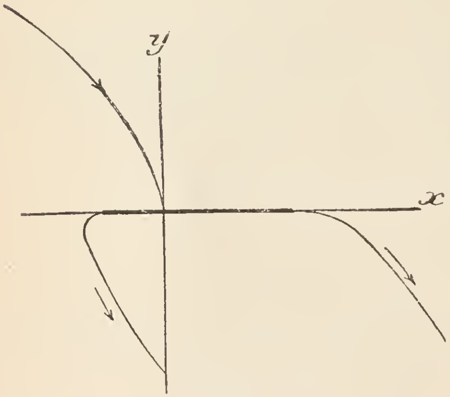


Fig. 7.

(III.) Still one more example may be introduced. Let there be

$$f(\omega)=\frac{k}{\sqrt{1-e^{-\omega}}}$$

and let $\psi$ vary between $-\pi$ and $+\pi$, but $\varphi$ between $-\infty$ and $+\infty$.

From the point $\omega=0$ draw a section for which $\psi=0$, and $\varphi>0$, and assume that for $\varphi=+0$, and $\psi=+0$, the real part of $f(\omega)$ is positive. The points of bifurcation of $\sqrt{f(\omega)f(\omega)-1}$ are the two points $\omega=0$, and $\omega=-\log(1-k^2)$ both which are found upon the section that has been drawn. The sign of the radical quantity $\sqrt{f(\omega)f(\omega)-1}$ is determined by the rule that its real part shall be positive for $\varphi=+0$, and $\psi=+0$.

Finally we assume that $\omega$ and $z$ disappear simultaneously.

At the boundary of the region of $z$ we have, first the line for which $\psi=0$, and $\varphi>0$. This line is composed of the following portions:

For $\psi=+0$, and $0<\varphi<-\log(1-k^2)$ we have,

$$x=\int_0^\phi\left(\frac{k}{\sqrt{1-e^{-\phi}}}+\sqrt{\frac{k^2}{1-e^{-\phi}}-1}\right)d\varphi$$

$$y=0$$

For $\psi=-0$ and $0<\varphi<-\log(1-k^2)$ we have,

$$x=-\int_0^\phi\left(\frac{k}{\sqrt{1-e^{-\phi}}}+\sqrt{\frac{k^2}{1-e^{-\phi}}-1}\right)d\varphi$$

$$y=0$$

These equations represent a portion of the axis of $x$ that is to be assumed as the fixed partition. Adjoining this fixed partition there comes as the free boundary of the moving fluid the line for which

$$\psi=+0,\ \varphi>-\log(1-k^2),$$

therefore

$$\frac{dx}{d\varphi}=\frac{h}{\sqrt{1-e^{-\phi}}};\ \frac{dy}{d\varphi}=-\sqrt{1-\frac{k^2}{1-e^{-\phi}}}\ :$$

and also the line for which

$$\psi=-0,\quad \varphi>-\log(1-k^2),$$

whence

$$\frac{dx}{d\varphi}=-\frac{k}{\sqrt{1-e^{-\phi}}};\ \frac{dy}{d\varphi}=-\sqrt{1-\frac{k^2}{1-e^{-\phi}}}\ .$$

The remaining boundaries of the region of $z$ are the lines

$$\psi=-\pi,\quad \psi=+\pi,\quad \varphi=-\infty,\quad \varphi=+\infty.$$

For $\psi=-\pi$ we have,

$$\frac{dx}{d\varphi}=-\frac{k}{\sqrt{1+e^{-\phi}}};\ \frac{dy}{d\varphi}=-\sqrt{1-\frac{k^2}{1+e^{-\phi}}}\ .$$

For $\psi=+\pi$ we have,

$$\frac{dx}{d\varphi}=\frac{k}{1+e^{-\phi}};\ \frac{dy}{d\varphi}=-\sqrt{1-\frac{k^2}{1+e^{-\phi}}}\ ;$$

These two stream lines are free boundaries throughout their whole extent.

For $\varphi=-\infty$, we have $\dfrac{dz}{dw}=-i$;

for $\varphi=+\infty$, and $\psi<0$, we have $\dfrac{dz}{dw}=-k-i\sqrt{1-k^2}$

and for $\varphi=+\infty$, and $\psi>0$, we have $\dfrac{dz}{dw}=k-i\sqrt{1-k^2}$

For $\varphi=-\infty$, we therefore have $y=+\infty$, and the stream flows with a velocity of 1 in the direction of the negative axis of $y$; for $\varphi=+\infty$, we have $x=\mp\infty$, and $y=-\infty$, and the stream flows with a velocity of 1 in a direction that makes an angle whose cosine is $\mp k$ with the direction of the positive axis of $x$.

In Fig. 8 the boundaries of the moving fluid are represented for this case.

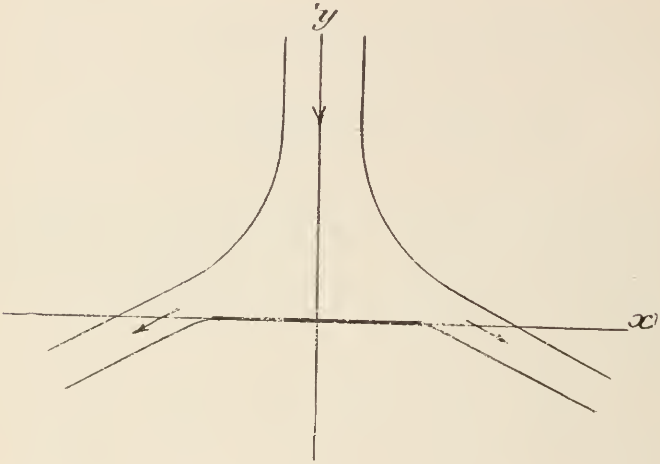



Fig. 8.

IX.

# DISCONTINUOUS MOTIONS IN LIQUIDS.*

By Prof. A. OBERBECK.

## I.

It is customary to designate by the term discontinuous fluid motions, those phenomena of movement in which the velocity is not throughout the whole space filled with the fluid a continuous function of the location. Therefore in such movements there occur surfaces within the fluid that separate from each other regions within which the velocities differ from each other by finite quantities. The fundamental principles of the theory of these motions were first given by Helmholtz.†

If we assume that a velocity potential ($\varphi$) does exist for so-called steady fluid motions then the hydro-dynamic differential equations can be summarized in the one equation,

$$p=C-\tfrac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}$$

Now Helmholtz has shown that the pressure $p$ and consequently the velocity can be discontinuous functions of the coordinates and that there are a great number of phenomena of motion for which the assumption of a discontinuous function is necessary. Especially has this theory been applied by Helmholtz and by Kirchhoff to fluid jets,‡ and the boundaries of free jets can be given under the following assumptions:

(*a*) That no accelerating force acts upon the fluid.

(*b*) That the movement is steady.

(*c*) That the movement depends only upon two variables, $x$ and $y$, and is therefore everywhere parallel to a fixed plane.

If in other cases, for instance for jets that are symmetrical about an axis or that are under the influence of the accelerating force of gravity,

---

* Read at the session of the Physical Society in Berlin, May 11, 1877. Translated from Wiedemann's *Annalen der Physik und Chemie*, 1877, vol. II, p. 1–16.

† See the *Berlin Monatsberichte*, 1868, p. 215 [or No. II of this series of Translations.]

‡ See Crelle's *Journal* vol. LXX, p. 289–299, [and Nos. III and VIII of this collection of Translations.]

it is not yet possible to determine the free boundaries by computation, then this is only because of the analytical difficulties. In general, however, one can judge of the nature of these boundaries from a consideration of the results already found.

The mathematical investigations just referred to hold good equally well for liquid jets that are bounded by quiet air as for those that are bounded by similar quiet liquid. In the actual production of such liquid jets it of course makes a great difference whether we allow water to flow into the air or water to flow into water. In both cases disturbing circumstances occur of which the mathematical theory takes no consideration. The jets of water projected freely into the air have been most thoroughly investigated.*

In these experiments the formation of jets occurs just as would be expected according to theory. On the other hand, however, it is known that water jets are influenced to an important extent by the capillary tension of the free surface, and that in consequence of this at certain distances from the orifice they break up into drops.

If we allow a liquid to flow into a similar quiet liquid then these capillary effects do not occur; but in place of this another disturbing cause, the viscosity, influences the phenomena. The viscosity has hitherto not been taken into consideration in the theory of the discontinuous movements of fluids. If we attempt to consider it we stumble upon a peculiar difficulty that has led the present author to experimentally investigate this class of fluid motions.

## II.

It is well known the theory of viscosity of fluids can be developed from the assumption first framed by Newton,† namely, that the retarding or accelerating influence of two portions of fluid flowing past each other with different velocities is proportional to their relative velocity. Especially has O. E. Meyer from this hypothesis developed the general differential equations for the motion of fluids.‡

If we assume that all parts of the moving fluid describe parallel paths, say in the direction of the axis of $y$, and that the velocities $v$ are only functions of $x$ and that finally $\mu$ is the coefficient of viscosity, then will the influence of two neighboring parts upon each other be represented by the expression

$$\pm \mu \frac{dv}{dx}.$$

---

* Besides the older experiments of Bidone and Savart see especially Magnus, Poggendorff *Annalen*, vols. XCV and CVI.

† Mathematical Principles of Natural Philosophy: German translation by Wolfers, Berlin, 1872, p. 368.

‡ See Crelle, *Journal*, vol. LIX, pp. 229–303, and Poggendorff *Annalen*, vol. CXIII, pp. 68, 69.

If $v$ is a discontinuous function of $x$, then at such a locality the differential quotient will be indefinitely large. Therefore two neighboring portions would exert an indefinitely great influence upon each other. If therefore one of the fluid portions is at rest while a neighboring portion that belongs to the jet flows by the first with a constant velocity communicated to it by some exterior influence, then the first or quiet particle must immediately begin to take part in the movement of the second, but the second on the other hand must begin to lose a definite fractional part of its velocity. The jet must therefore rapidly set the surrounding quiet fluid in motion with it. It would according to this appear to be doubtful whether sharply defined jets such as are demanded by the above-mentioned theory of Helmholtz could be formed in a fluid subject to viscosity.

The few experiments made hitherto upon this question appear to confirm this suspicion. Especially notable is an investigation by Magnus (Poggendorff *Annalen*, LXXX, pp. 1–40), who allowed pure water to flow from a cylindrical opening into a weak solution of salt and, by means of a glass tube drawn out into a fine point, led away a small quantity of the inflowing water in the neighborhood of the opening. The liquid thus caught was examined as to its salinity. From the latter one could calculate to what extent the inflowing liquid had become mixed with that which was previously in the vessel. It resulted that pure water could not be caught at any point of the inflowing liquid; that therefore everywhere the original quiet liquid was carried along with the moving liquid.

The analogous case of jets of air and of smoke, as also that of the free jets of water in the air, demonstrates that in all these, we have to do with phenonena of very slight stability. It is well known how sensitive such jets frequently are with respect to the feeble periodic disturbances produced by waves of sound.*

It seemed to me therefore of interest to investigate more accurately the formation of water jets in water and therein to utilize a method that allows of following the course of the phenomena of motion better than was possible in the experiments of Magnus. This object is most simply attained in that we allow feebly tinted water to flow into colorless water. Fuchsin is used as coloring material. It is well known that with a very small quantity of this material an intense red color is produced with no fear lest hereby the specific gravity of the water be essentially changed. In the first experiments performed with this it resulted that the jet of colored liquid broke up at a very slight distance from the orifice into reddish clouds and drops that mixed with the quiet liquid and carried it along with them. By further investigation however, it became possible to determine conditions under which real jets of considerable length and sharp boundaries were formed. These were of great sta-

* See John Tyndall on Sound, pp. 289–292 of the German edition edited by Helmholtz and Wiedemann, Brunswick, 1869.

bility, so that small disturbances had only a rapidly diminishing influence upon their course. At the forward end of these jets there formed peculiar surfaces of flow that plainly allowed the influence of viscosity to be seen. These phenomena of motion are of remarkable beauty and delicacy, of which any one may convince himself who performs the easily repeated experiments.

Since the theoretical investigations mentioned in the introduction treat of the modifications of jets by solid bodies, and Kirchhoff especially gives a series of interesting examples bearing upon this, therefore this question has also been taken into consideration in my experiments. Very stable forms of jets are also thus formed that have more similarity with those deduced by theory than one could have expected.

## III.

The experiments were made with the following simple apparatus: A cylindrical glass vessel (Fig. 9), of about 60 centimetres height and 12 centimetres diameter, was filled with water. Into this there passed a flow of water from a filter through an India-rubber tube, a glass stop-cock, and a glass tube. The filter, as also the entire tubular system, was filled with the colored liquid. After filling with water the glass cylinder (in whose place one may also use any large glass vessel), one must wait a long time until the motion of the water has been destroyed by viscosity. The experiment succeeds best when the water has stood for many hours in the cylinder, since then currents resulting from differences of temperature are no longer present. By a quick opening of the glass stop-cock one can allow a definite quantity of colored liquid to enter into the quiet liquid, or by a longer opening one can attain a steady stationary current. By elevating or depressing the filter one can easily regulate the height of the upper fluid level. The use of a small difference of pressure was found to be the principal condition for the maintenance of regular current formations.

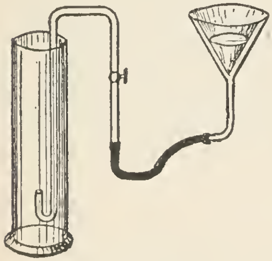


Fig. 9.

The majority of the experiments if no other problem was on hand were executed with an excess of pressure of about 20 millimeters. By means of proper arrangements solid bodies could be opposed above the jet. For exact observation it is necessary to fasten a surface of white paper behind the glass cylinder.

## IV.

In order to understand the formation of jets it is advantageous first to learn the behavior of a definite quantity of liquid entering under

a small excess of pressure into the quiescent liquid. I therefore begin with a description of the experiments relative to this.

If we allow the stop-cock to be opened for only a short time, then even with the smallest differences of pressure of two or three millimetres, a sharply defined mass of liquid penetrates into the quiescent liquid. The original form of this mass is soon modified by viscosity and by the participation of the hitherto quiet liquid in its motion, in a peculiar manner, and finally it rolls itself into a ring. The colored mass of liquid goes through the series of forms presented in Figs. 10, 11, 12, and 13. Of these drawings, as of most of the following ones, it is to be noted that they represent a section of the mass of liquid by a plane that passes through the axis of symmetry of the formation. In order to find the true form therefore, one must imagine the figure revolved about this axis.

Fig. 10.


Fig. 11.

Fig. 12.

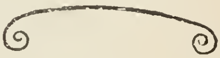


Fig. 13.

With the form of Fig. 13, the ring formation is completed. Moreover in general even for differences of pressure of 10 to 20 millimetres, the living force of the liquid was consumed so that this figure long floated motionless in the colorless liquid.

If we use somewhat larger differences of pressure we observe that the liquid within the ring continues rotating for a longer time. The original progressive movement has therefore been transformed into a vortex movement. The vortex movements have been theoretically treated by Helmholtz* and he has in the introduction to his memoir referred to the necessity of the transformation of any current or movement that has a velocity potential into a vortex movement in consequence of viscosity.

Many other consequences drawn by Helmholtz in his memoir just referred to can be easily observed by the help of the apparatus used by the present writer.

*Crelle's *Journal* LV, pp. 25–56, [and No. II of this collection of Translations.]

If by alternately opening and closing the stop-cock we allow two drops to enter into the colorless liquid in rapid succession, then there arises a ring formation for each drop and the following one always catches up with the preceding one. Different cases are then possible, according to the differences of pressure that are used; if these are slight then the second ring is not able to penetrate the first one and a formation, as shown in Fig. 14, remains for a long time visible in the fluid. With greater differences of pressure, on the other hand, ring No. 2 passes through ring No. 1, since the former contracts while the latter expands. One can then observe that afterwards ring No. 1 endeavors on its part to pass through ring No. 2. But generally the living force is by this time consumed, so that ordinarily the two rings settle into the formation shown in Fig. 15. This interchanging passage

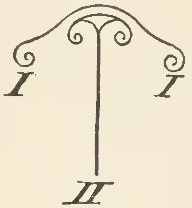



Fig. 14.

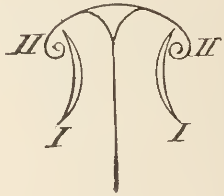



Fig. 15.

of the vortex rings through each other was predicted by Helmholtz from theory in the memoir above referred to.

Reusch has occupied himself experimentally with the formation of vortex rings.* After having described in detail the formation of smoke rings in the air, he passes to the formation of rings by the sudden entrance of a small quantity of colored liquid into colorless liquid. Although in his arrangement of the experiments the transition of the progressive into the vortex motion is very rapidly completed, still he also has frequently observed the intermediate stages shown in Figs. 11 and 12 and described them very appropriately as mushroom formations. The manner of this transition is seen directly from the examination of Figs. 10 to 13. Evidently there arise two currents in the quiescent liquid. The one current, indicated by the arrows $A$ and $B$, is produced by the progressive movement of the drop, which moves forward in the liquid almost as a solid body. The other current, in the direction of the arrows $C$ and $D$, is principally produced by viscosity. The formation of the spiral surface of rotation is finally the necessary consequence of these two opposite currents.

* Poggendorff's *Annalen*, vol. cx, pp. 309–316.

V.

We can now pass on to the formation of jets proper by steady currents. If we allow the stop-cock to be open for a long time there arises (at first rapidly, afterward slowly) a jet whose upper portion has great similarity with the forms hitherto described. The jet soon attains a certain altitude that depends upon the difference of pressure and above which it does not ordinarily go, or at least only with extreme slowness. Thus for a difference of pressure of 5 millimetres the altitude of the jet is about 20 millimetres; for 10 millimetres pressure the altitude is about 80 millimetres; for 20 millimetres pressure the altitude is 200 millimetres; and for 30 millimetres difference of pressure the jet attains the upper limit of the water at an altitude of about 400 millimetres in about 80 seconds. The colored liquid spreads out over the surface of the colorless water and thence diffuses very slowly downward. The above given numbers do not present any general law, but only give approximately the connection between the altitude of the jet and the difference of pressure. The former also depends somewhat on the specific gravity of the inflowing liquid, which varies a little with the quantity of added coloring material. It depends also on the size of the discharging aperture. Moreover the form of the front part of the jet is not always exactly the same; in the figs. 16 and 17 are given two of the ordinary forms of jet. In both these forms the jets proper are the same; the bell-shaped expansion, however, is formed in a somewhat different manner, perhaps conditioned by small variations of temperature in the colorless liquid.

Fig. 16.

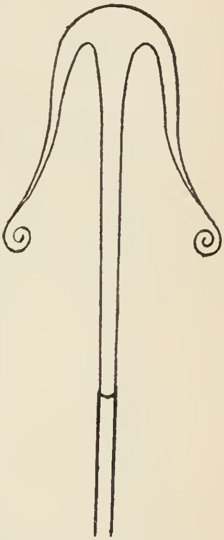


Fig. 17.

By the avoidance of all disturbances the jets here described remain many minutes entirely unchanged. Only the bell-shaped portion continues to extend slowly somewhat further downwards. Moreover, with respect to small disturbances the jets showed themselves by no means very sensitive. If by a gentle pressure on the India-rubber tube the velocity of the discharging liquid is diminished for an instant, then water presses from all sides into the jet; after the cessation of this pressure the original form of the jet is immediately resumed. Even

when the pressure on the rubber tube is periodically increased and diminished for a long time the continuity of the jet is not completely broken. Such a jet presents a very remarkable appearance, which is reproduced as well as possible in Fig. 18.

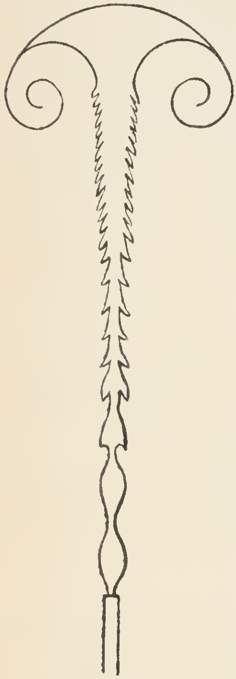


Fig. 18.

The phenomena hitherto described occur with differences of pressure of 60 millimetres at the maximum, but very different results are obtained if larger differences of pressure are used. With 80 or 90 millimetres we obtain jets of the greatest sensitiveness. By every small disturbance the continuity of the jet is broken, and it must then form for itself a new path every time. Above 100 millimetres difference of pressure there are formed only very short jets in the immediate neighborhood of the opening. These at a slight altitude break up into a cloud of individual small drops that under the rapid motion immediately mix with the colorless liquid.

When (as an experiment) colored liquids were used whose specific gravity differed considerably from that of the colorless water, no regular discontinuous currents could be obtained; thus in one experiment a solution of salt was added to the colored water, in another experiment some alcohol was added. The salt solution immediately after its discharge fell in thick, irregular, heavy drops back upon the discharge pipe, while the alcohol moved in very thin threads, frequently broken up, toward the upper free surface of the water.

From the experiments hitherto described it follows that in fact steady jets form with small differences of pressure. The viscosity therefore does not prevent discontinuous currents. Viscosity appears in general to exert so unimportant an influence upon the cylindrical portion of the jet formation that we are tempted to assume the real possibility of the sliding of moving particles of water past those at rest, as the simpler theory assumes to be the fact, without any consideration of viscosity. If, however, the transition from the finite velocity of the jet to the quiet fluid does not take place within the thickness of a mathematical cylindrical surface but gradually within a layer of a definite thickness then this thickness can only be extraordinarily small and appears not to change with the time. That on the other hand the viscosity plays an important rôle in the origin of the jets is already mentioned above. The principal proof of this consists in the invariable formation of spiral surfaces of rotation into which the jet is transformed. The origin of these assumes that the colorless liquid in the neighborhood of the jet receives a certain velocity in the direction of the jet,

The greater sensitiveness of the jets for large velocities of current, as also the impossibility of forming alcohol jets in water, is a direct consequence from the theory of discontinuous fluid motions. Since the difference of pressure in the moving and the quiet fluid is proportional to the square of the velocity, therefore for greater velocities the quiet liquid presses directly into the jet as soon as a slight disturbance occurs in its uniform course. When finally, with more rapid outflow, such disturbances occur continually, then in general a jet can not form.

VI.

As already remarked above it is of interest to know the path that a jet will describe when it meets a solid body in its path. The bodies used for this purpose by me were of different kinds, and by a simple arrangement were brought in the neighborhood of the discharging aperture before the jet was produced by opening the stopcock. It is of course understood that at the beginning of the experiment one waited a long time until the movements of the liquid caused by these operations had subsided. Equally also was the solid body first freed from the air bubbles that adhered to it.

The processes that occurred are most easily seen by considering the following experiment. If the jet strikes upon the sharp edge of a thin sheet of iron that passes parallel to the direction of the jet and through its axis, then it is cut into two portions which are deviated from the vertical direction of the current. The angle between these side currents and the original direction of the jet becomes smaller little by little. The cause of this phenomenon consists in the fact that not only the solid body but also the fluid attached thereto force the moving fluid into a departure to one side. With currents of longer duration on the other hand a part of the quiet liquid is carried along so that the two upper branches of the jet slowly change their direction of motion and more and more nearly approach the plane of the sheet-iron. Still one can always observe quiet colorless liquid between the moving colored liquid and the sheet-iron. The progress of this phenomenon depends upon the original difference of pressure or correspondingly upon the velocity of the flowing liquid. For a small velocity the current flows as shown in Fig. 19; for greater velocities, on the other hand, the two portions of the jet after a time take the position shown in Fig. 20, where the dotted part of the figure is intended to show the initial direction of the current.


Fig. 19.

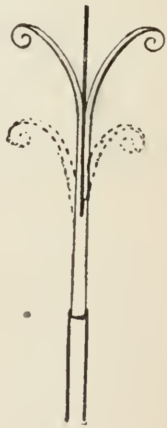


Fig. 20.

The peculiar behavior of the originally adherent quiescent liquid, which is afterward carried along, explains the slow changes in the path of the current that is also observed with other solid bodies of different shapes.

If a jet strikes upon a small brass sphere then with steady flow the stream path gradually takes the form shown in Figs. 21, 22, 23, and 24.

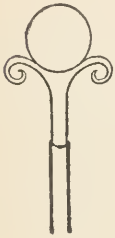


Fig. 21.

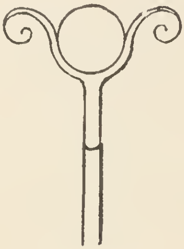


Fig. 22.

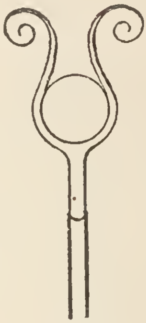


Fig. 23.

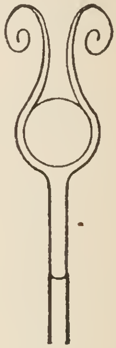


Fig. 24.

We see how at first the sphere and the adherent liquid force the moving liquid to a deviation almost at right angles from its original course. Then gradually the quiescent liquid is carried along, the stream surface follows the surface of the sphere continually more and more closely. A consideration of the thin stream surface that finally encloses the greater part of the sphere tempts one to assume that the moving fluid glides along the surface of the sphere. At least, by means of small solid bodies occasionally occurring in the liquid, one recognizes that in the immediate neighborhood of the fixed obstacle the liquid moves with finite velocity.

The phenomena just described do not appear to depend especially on the substance of the solid body, assuming of course that it is provided with a smooth surface. Instead of the brass sphere an ivory sphere may be used. This is in the same way gradually covered over with a close-fitting stream surface. Similar to this was the process when the jet struck against the lower end of a test tube. With a steady current the lower part of the tube is slowly covered over with a thin stream surface, which at a distance of about 4 centimetres from the lower end of the glass surface bent away and ran into the spirals that here also perpetually recur.

Of further special interest is the case where the jet meets a definite thin partition perpendicular to its own direction, since this current has been theoretically treated by Kirchhoff (Crelle's *Journal*, vol. LXX, p. 298), but under the rather different conditions already mentioned. Therefore, a small circular plate was placed perpendicular to the jet. The stream lines in this case depend essentially on the ratio of the radii of the plate and the jet. If the radius of the circular plate is

materially larger than that of the jet, then the latter will be deviated at the plate through a right angle and flows in a thin layer radially along the plate, which it leaves in a horizontal direction, as in Fig. 25.

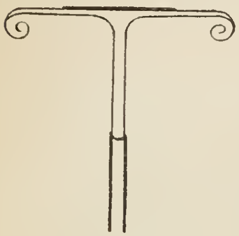


Fig. 25.

If, on the other hand, the radius of the partition is only a little larger than the radius of the jet, then will the stream lines be deviated by a smaller angle from their original direction. This process is shown in Fig. 26, which has a great similarity to the drawing given by Kirchhoff at the place just refered to.*

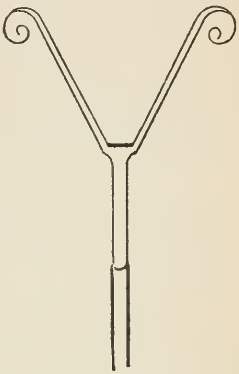


Fig. 36.

A thin sharp-edged partition that extends to about the center of the jet exerts a very similar influence to the thin circular plate. In this case, while one part of the jet spreads in a thin layer along the plate the other part is deviated through an acute angle. In this experiment also the material of the plate appears to exert no sensible influence on the course of the stream. Disks of thin glass and of glazed drawing paper were used, while the above-mentioned thin partition was replaced by a sheet of tinfoil, which was stretched over a glass frame, and one-half of which had been removed along a straight line. The stream phenomena remained exactly the same. The angle by which the jet in this last case was deviated from its initial direction depended principally upon the depth to which the thin partition penetrated into the jet.

The phenomena here described of flow against solid bodies succeeded only for small velocities of the jet such as corresponded to differences of pressure of 20 or 30 millimetres.

## VII.

Since it was the main object of the author to investigate discontinuous liquid motions in their simplest form, therefore he has for the present confined himself at first to the above-described experiments. Still these shall be extended as soon as possible in different directions. As the next points for study the following especially commend themselves:

(*a*) The flow of a colored liquid into a colorless one through an opening in a thin partition. Some preliminary experiments with an imperfect apparatus showed that the jets thus formed are similar to the above described under otherwise similar circumstances.

(*b*) The discharge of a liquid into another liquid of equal specific gravity that is not miscible with the first liquid. In this experiment

* [See No. VIII of this collection of Translations.]

one could make use of the liquids employed by Plateau, namely, oil and alcohol of equal specific gravity. The question will here arise, in what manner the formation of a jet is modified by capillary action.

(*c*) A stream of air in moving air, the latter being made visible by smoke.

VIII.

The results of the present investigation can be summarized in the following theorems:

(*a*) The viscosity of fluids does not prevent the formation of steady discontinuous fluid motions. In consequence of friction these motions in the beginning suffer important modifications by reason of simultaneous spiral movements; but with long continued flow they form sharply defined fluid jets.

(*b*) The jets thus formed are very stable for small velocities, and even after small perturbations again immediately assume their original form. For greater velocities, on the other hand, they become very sensitive. If the velocity exceeds a certain limiting value, then only very short jets form in the immediate neighborhood of the opening.

(*c*) The jets are not only modified in their movement by solid bodies but also by the liquid adhering to these. The latter adherent liquid is slowly pushed aside by the jet. If then the body is bounded by a continuously curved surface the flowing liquid surrounds it in a thin layer. If, on the other hand, the solid body is bounded by a surface that at certain points has an indefinitely large curvature, such as a sharp edge, then the stream lines follow it only up to this edge and from that point on leave the solid body.

(*d*) The theory of discontinuous fluid motions, as Helmholtz and Kirchhoff have thus far developed it [for perfect fluids], also gives in general the phenomena observed in a fluid subject to friction. The only difference is the formation of vortex motions simultaneous with origination of the jets.

In conclusion we may call attention to the fact that in nature we find a whole series of processes that have a common origin with those just described. These are to be observed in the currents in rivers and canals, especially at places where the banks have sharp corners or where solid bodies, like the piers of a bridge, retard the uniform movement. The eddying motions there occurring clearly show where the quiet and the moving liquids border on each other. Since as a specially noteworthy result of the investigation here communicated has been to show that discontinuous motions arise even for very small differences of pressure, therefore it is easy to see that they must occur often enough in the last mentioned streams.

X.

## THE MOVEMENTS OF THE ATMOSPHERE ON THE EARTH'S SURFACE.*

By A. OBERBECK

### I. INTRODUCTION.

The investigations of Guldberg and Mohn† on the motions of the atmosphere certainly occupy a prominent place in the development of theoretical meteorology. If not the first they are at least the most extensive and successful attempt to explain the most important phenomena of the motion of the air by the principles and fundamental equations of hydrodynamics. I would especially indicate as the special service of the authors that they have brought the problem of the motions of the air into a form amenable to mathematical treatment by simple but as I believe thoroughly appropriate assumptions. They themselves have already computed a series of interesting atmospheric movements that frequently occur in nature, especially the cases where the isobaric systems consist of parallel straight lines or concentric circles.

I have attempted in the present work to go further on in the path laid out by Guldberg and Mohn, especially in that I have endeavored to apply to the atmosphere the methods developed in hydrodynamics for other problems.

In the present memoir the steady movements of the atmosphere, or, as Guldberg and Mohn call them, "invariable systems of winds," are principally treated. It is natural to refer the movements of the atmosphere back to the general modes of motion of fluids, that is to say, to motions that are characterized by a velocity potential and to vortex motions. In this way it is possible to attain solutions of great generality that can be applied to any system of isobars whatever. By this method of treatment it is further possible to overcome a difficulty that occurs in the theory of cyclones of Guldberg and Mohn, without as it would appear having been hitherto observed. These investigators distinguish correctly an inner and an outer region for each cyclone, in

* Translated from Wiedemann's *Annalen der Physik und Chemie*, 1882, vol. XVII., pp. 128–148.

† "Studies on the Motions of the Atmosphere." Christiania, Part I., 1876, Part II., 1880.

which the expressions for the velocity of the air at the earth's surface and for the pressure follow different laws. But they have not attempted so to deduce the expressions for the velocity and for the pressure in these two regions from one common principle, that these velocities and pressures merge into each other continuously at the boundary. In the computation of numerical examples they have sought to help over this difficulty by not applying their formula to the region in the neighborhood of the boundary, but have here by interpolation introduced numerical values passably good, but therefore certainly rather arbitrary. Above all however it is a serious matter that according to their theory the direction of the wind at the boundary suddenly varies through a definite angle. The want of continuity here spoken of can originate either in the assumptions adopted as a basis or in the execution of the computation. I have arrived at the conviction that the latter is the case.

I have therefore started from the same assumptions as Guldberg and Mohn; these are given in the following Section (II) and I add only thereto the following principle, about which there can be no doubt: "*The pressure of the air, as also the velocity of the air and its direction, ought to experience only continuous variations throughout the whole region under consideration.*"

By the application of this fundamental principle the theory of cyclones, even in the case of circular isobars, deviates not a little from the theory established by Guldberg and Mohn.

## II. ASSUMPTIONS THAT ARE THE BASIS OF THE PRESENT TREATMENT.

The following assumptions form the foundation of my mathematical development:

(*a*) The portion of the earth's surface coming into consideration is assumed to be a plane. A constant average value will be assumed for the geographical latitude of this region.

(*b*) The air will be treated as an incompressible fluid.

(*c*) The investigation here carried out refers only to a stratum of air of moderate height above the earth's surface. The latter surface exerts a retarding influence on the movements of the air that can be considered as a force opposed to the movement and proportional to the velocity.

(*d*) The currents of air at the earth's surface are ordinarily directed toward a center or flow away from the neighborhood of such a center. Such currents can not be imagined without the existence of a vertical current in their neighborhood. If, therefore, we in general confine ourselves to the consideration of horizontal currents, still the consideration of vertical motion is not to be avoided for the neighborhood of such a center. We have, therefore, to distinguish between regions of pure horizontal motion and regions with vertical motion. As to the latter,

the following simple assumption is made:* If we adopt a system of rectangular coördinates such that the plane of $x\,y$ is the horizontal plane while the axis of $z$ is directed vertically upward, then for the vertical component of an ascending current of air we have the expression

$$w=c.z.$$

If the boundary of the region above which this current ascends is known, while outside this boundary the movement is exclusively horizontal, then the whole system of winds (the cyclone) is thereby completely determined. The quantity $c$ can be designated as the constant of the ascending air. For regions with descending air currents the negative sign must be given to the constant.

The region for which

$$w=c.z$$

will, for brevity, be designated as the inner region of the cyclone; that for which

$$w=0$$

will be designated as the outer region.

The vertical component is to be considered only in connection with the equation of continuity. Therefore for the outer region this equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

and for the inner region

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b).$$

For both regions, moreover, the ordinary equations of hydro-dynamics for movements in one plane hold good, namely:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=X-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=Y-\frac{1}{\rho}\frac{\partial p}{\partial y},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the letters have the ordinary signification.

The accelerating forces whose components are $X$ and $Y$ must express the influence of the rotation of the earth and of friction.

The consideration of the earth's rotation necessitates the introduction of a force† whose components are

$$X_1=-\lambda v,\ Y_1=+\lambda u.$$

---

*This agrees entirely with the assumption of Guldberg and Mohn as to the vertical currents. (See *Études*, 1876, part 1, p. 28.)

†See G. Kirchhoff, *Vorlesungen über Mechanik*, Leipzig, 1876, pp. 87–95.

In these we have put

$$\lambda = 2\sigma \sin \beta,$$

wherein $\sigma$ is the angular velocity of the earth (0.00007292) and $\beta$ the mean geographical latitude of the region in question. Herein the system of coördinates is to be so taken that the resultant produces in the northern hemisphere a deviation of the path toward the right. Therefore the axis of $X$ is positive toward the east and the axis of $Y$ positive toward the south. For the resistance of friction, according to the adopted assumption ($c$), we put

$$X_2 = -ku, \quad Y_2 = -k\,v.$$

The factor $k$ is dependent on the nature of the earth's surface. It is smaller for the surface of the ocean than for that of the land and is of the same order of magnitude as $\lambda$.

By the introduction of these forces in the equations of motion (2) we obtain

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -ku - \lambda v - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -kv + \lambda u - \frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3).$$

If after the addition of $\pm v\,(\partial v / \partial x)$ to the first equation and of $\pm u\,(\partial u / \partial y)$ to the second we introduce the double angular velocity $\zeta$ in reference to the axis of $Z$, so that

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

then equations (3) can be written in the form

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left\{\frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2)\right\} + \frac{\partial u}{\partial t} + ku &= -(\lambda + \zeta)\,v \\ \frac{\partial}{\partial y}\left\{\frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2)\right\} + \frac{\partial v}{\partial t} + kv &= \phantom{-}(\lambda + \zeta)\,u \end{aligned}\right\} \quad . \quad . \quad (5).$$

## III. DEDUCTIONS FROM THE FUNDAMENTAL EQUATIONS AND THEIR TRANSFORMATION.

From equations (5) we can deduce without further special assumption a theorem that expresses a general relation between the gradient, the wind velocity and the wind direction. As is well known in meteorology, the term gradient indicates the difference in the atmospheric pressure at two localities that lie at a definite distance apart in the direction of the most rapid change of pressure. According to this we can consider the differential quotient

$$\frac{1}{\rho}\frac{dp}{dn}$$

as the analytical expression for this quantity, omitting a factor that depends upon the adopted units, and in which $dn$ is an element of the normal to the curve whose equation is $p/\rho$=constant.

I put

$$\gamma = \frac{1}{\rho}\frac{dp}{dn} = \frac{1}{\rho}\sqrt{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2}$$

Furthermore, let the velocity of the wind at a point $x\ y$ be $\omega$ so that $\omega^2 = u^2 + v^2$ and let $\varepsilon$ be the angle between $\omega$ and $\gamma$, in which $\gamma$ must indicate the direction of diminishing pressure. Then we have

$$\cos\varepsilon = -\frac{u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}}{\omega\sqrt{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$\omega\,\gamma.\cos\varepsilon = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we multiply the first of equation (5) by $u$ and the second by $v$ and add together, there results,

$$\omega.\,\gamma.\cos\varepsilon = k\omega^2 + u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + \frac{1}{2}\left\{u\left(\frac{\partial\omega}{\partial x}\right)^2 + v\left(\frac{\partial\omega}{\partial y}\right)^2\right\}$$

or

$$\gamma\cos\varepsilon = k\omega + \frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}$$

for which by introducing the notation

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}$$

we can write

$$\gamma\cos\varepsilon = k\omega + \frac{d\omega}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From these equations many consequences can be drawn that lead to specially simple theorems when the velocities of the wind are so small that the term

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}$$

can be neglected. But the following theorems will also be approximately true even if the velocities are larger.

($a$) If we compare an invariable system of wind and one that is variable as to its intensity, and of which we will assume that at any given instant there prevails throughout it everywhere uniform velocities and

uniform gradients, then the angle between the direction of the wind and the gradient is smaller in the variable system than in the invariable when the intensities increase, and, inversely, larger when the intensities are diminishing.

(*b*) If in one and the same system of winds having a progressive movement we compare two points that have equal velocities and equal gradients then the deviation of the wind direction from the gradient is smaller at the point where the wind velocity is increased than where it is diminished. Therefore in general the departures from the gradient will be smaller throughout the advancing half or front of a moving cyclone than within the rear half.

(*c*) For steady motions of moderate intensity, where therefore

$$\frac{d\omega}{dt}=0,$$

the velocity is proportional to the projection of the gradient on the direction of movement. Furthermore for equal gradients and equal velocity the deviation of the direction of the wind is greater in proportion as the friction is less.

Some of these theorems have been proven already for special cases by Guldberg and Mohn. The theorem expressed in paragraph (*c*) has also been attained in an entirely different way by A. Sprung.*

I pass now to the investigation of the invariable systems of wind, and therefore assume that

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=0.$$

If further we put

$$P=\frac{p}{\rho}+\frac{1}{2}(u^2+v^2) \tag{9}$$

then the equations (5) give

$$\left.\begin{aligned}\frac{\partial P}{\partial x}+ku&=-(\lambda+\zeta)v;\\ \frac{\partial P}{\partial y}+kv&=+(\lambda+\zeta)u\end{aligned}\right\}(10).$$

If in these we introduce for $u$ and $v$ expressions of the form ordinarily used in hydro-dynamics, namely:

$$\left.u=\frac{\partial \varphi}{\partial x}+\frac{\partial W}{\partial y}, \qquad v=\frac{\partial \varphi}{\partial y}-\frac{\partial W}{\partial x}\right\}(11)$$

* See Wiedemann, *Beiblatter*, 1881, vol. v, page 240: and Sprung *Meteorologie*, Hamburg, 1885.

and furthermore put

$$\left.\begin{aligned} f_1 &= P + k\varphi - \lambda W; \\ f_2 &= kW + \lambda\varphi \end{aligned}\right\} \quad (12)$$

we thus obtain

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} &= -\zeta\left(\frac{\partial \varphi}{\partial y} - \frac{\partial W}{\partial x}\right) \\ \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} &= +\zeta\left(\frac{\partial \varphi}{\partial x} + \frac{\partial W}{\partial y}\right) \end{aligned}\right\} \quad (13)$$

According to the equations (1$a$) (1$b$) and (4) the functions $\varphi$ and $W$ must for the outer region satisfy the partial differential equations

$$\varDelta\varphi = 0 \quad \text{. . . . . . . . .} \quad (14a)$$

but for the inner region the equation

$$\varDelta\varphi = -c \quad \text{. . . . . . . . .} \quad (14b)$$

and for both regions

$$\varDelta W = \zeta \quad \text{. . . . . . . . .} \quad (15)$$

where we have used the abbreviation $\varDelta$ for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For regions of pure horizontal motions solutions of these equations can be given of great generality, which will now be separately treated of.

## IV. ATMOSPHERIC CURRENTS IN REGIONS OF PURE HORIZONTAL MOTION.

When in accord with the assumption of purely horizontal motions we have $\varDelta\varphi = 0$ throughout the whole region under consideration, then we can also put $\zeta = 0$. In this case we can satisfy equation (13) if we put

$$f_1 = \text{constant}, \quad f_2 = \text{constant}.$$

The second of these equations gives

$$W = -\frac{\lambda}{k}\varphi \quad \text{. . . . . . . . .} \quad (16)$$

in which an arbitrary constant can be omitted. Then from the first of these equations, namely, for $f_1$, there results

$$P = \text{constant} - k\varphi\left(1 + \frac{\lambda^2}{k^2}\right) \quad \text{. . . . . .} \quad (17)$$

The component velocities are:

$$u=\frac{\partial\varphi}{\partial x}-\frac{\lambda}{k}\frac{\partial\varphi}{\partial y},\quad v=\frac{\partial\varphi}{\partial y}+\frac{\lambda}{k}\frac{\partial\varphi}{\partial x}\quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\omega^2=u^2+v^2=\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right\}\left(1+\frac{\lambda^2}{k^2}\right)\quad . \quad . \quad . \quad (19)$$

Finally, from equation (17) we obtain

$$\frac{p}{\rho}=\text{constant}-\left(1+\frac{\lambda^2}{k^2}\right)\left\{k\varphi+\tfrac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\right\}.\quad . \quad (20)$$

All these expressions still contain the as yet undetermined function $\varphi$, which is only limited by the condition $\Delta\varphi=0$. Such functions can be easily found in various ways. Thus if we bring the function of a complex variable $x+iy$ into the form

$$F(x+iy)=\varphi+i\psi,$$

then both $\varphi$ and also $\psi$ satisfy the above given differential equations. Moreover, both functions stand in the following relations to each other.

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y};\quad \frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

With the assistance of these equations one can easily find the general equation for the path of the wind. We obtain this from the differential equations

$$u\,dy=v\,dx,$$

$$\frac{\lambda}{k}\left\{\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy\right\}=\frac{\partial\varphi}{\partial x}dy-\frac{\partial\varphi}{\partial y}dx.$$

If we introduce $\psi$ into the right-hand side of this equation we obtain as the equation for the path described by the wind

$$\psi-\frac{\lambda}{k}\varphi=\text{constant}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The path of the wind intersects the system of lines defined by the condition $\varphi=$ constant at an angle that is everywhere the same. If we designate by $\varepsilon$ the angle that the direction of the wind makes with the normal to the curves $\varphi=$ constant then we have

$$\tan\varepsilon=\frac{\lambda}{k}.$$

For currents of air of moderate velocity the term

$$\tfrac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$

in equation (20) can be neglected in comparison with $\varphi$. In this case the isobars, for which $p$ equals a constant, are identical with the curves $\varphi=$ constant and we obtain the following general theorem;

*In regions of pure horizontal motion, and for moderate wind velocity, the angle between the direction of the wind and the gradient is constant and depends only on the constant of rotation and the constant of friction and is independent of the direction of the isobars.*

The above given relation had been found by Guldberg and Mohn * for the special cases of rectilinear and circular isobars.

The general solutions contained in equations (18, 19, and 20) can now be so applied that we may adapt the function $\varphi$ to any other given system of isobars. When this is achieved, then the motions of the air are determined by the first two of these equations.

If, for instance, we have to do with a region that is under the influence of numerous but distant maxima and minima of pressure, then we can approximately put

$$\varphi = \Sigma\, c. \log \rho.$$

In this expression $\rho$ indicates the distance of the point $(xy)$ from the vertical currents of the individual regions, assuming that the dimensions of these regions are small in comparison with the distances. This value of $\varphi$ would be exactly correct if all inner regions [namely, as defined on page 153] were bounded by circles. Then $\rho$ would indicate the distance from the center of the circle. The constants $c$ depend upon the intensity of the respective vertical currents. They are positive for the minima and negative for the maxima [*i. e.*, for areas of low and high pressure respectively]. The assumption

$$F(x+iy) = (x+iy)^2 = \varphi + i\psi$$

whence

$$\varphi = x^2 - y^2\,;$$

$$\psi = 2xy$$

leads to a special example already treated of by Guldberg and Mohn.†

The potential curves

$$x^2 - y^2 = \text{constant}$$

and the stream lines

$$2xy - \frac{\lambda}{k}(x^2 - y^2) = \text{constant}$$

are systems of equilateral hyperbolas.

---

* See their *Études*, etc., Part I, pp. 23-26.

† *Études*, Part II, pp. 51, 52.

If we assume

$$F(x+iy)=\log(x+iy)=\varphi+i\psi$$

and if we substitute

$$x=r\cos\theta;\ y=r\sin\theta$$

then follows

$$\varphi=\log r,\ \psi=\theta.$$

In this case the isobars consist of concentric circles. The paths of the wind are logarithmic spirals having the equation

$$\theta-\frac{\lambda}{k}\log r=\text{constant.}$$

## V. STEADY SYSTEMS OF WINDS.

It is certainly at present generally assumed in meteorology that the winds at the earth's surface owe their origin and maintenance to vertical currents of air that are limited to definite regions. Let us assume that there is given such a region having any arbitrary boundary above which a current of air ascends whose velocity in the neighborhood of the earth's surface is determined by the constant ($c$). By this assumption the whole system of winds dependent thereon, as well as the distribution of pressure, is determined for the whole region. It is therefore the province of mathematics to determine all the quantities coming into consideration both for the inner and also for outer region.

To this end the functions $\varphi$ and $w$ are to be propely determined. The first of these is found without further difficulty from well-known theorems in the theory of the potential. Since these functions must in the outer region satisfy the partial differential equation $\Delta\varphi=0$, and in the inner region must satisfy the equation $\Delta\varphi=-c$; therefore*

$$\varphi=-\frac{c}{2\pi}\int d\,\sigma\log\rho \quad \text{.....................(22)}$$

In this $\rho$ indicates the distance of the element of the surface $d\,\sigma$ from the point $x$, $y$. The integral is to be extended over the whole of the given inner region. Therefore the velocity potential is the logarithmic potential of a layer of matter having the density $-c/2\pi$ that covers the region of the ascending current of air. The function $\varphi$ itself, as also its first differential quotient, varies continuously throughout the whole plane up to the boundaries of the outer and inner regions.

*See G. Kirchhoff, *Vorlesungen über Mechanik*, 1876, p. 195.

Therefore, the function $W$ is known for the outer region and is

$$W=-\frac{\lambda}{k}\varphi.$$

In order to determine this function for the inner region also one must go back to the equations (13)

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}=-\zeta\left(\frac{\partial \varphi}{\partial y}-\frac{\partial W}{\partial x}\right)$$

$$\frac{\partial f_1}{\partial y}-\frac{\partial f_2}{\partial x}=+\zeta\left(\frac{\partial \varphi}{\partial x}+\frac{\partial W}{\partial y}\right)$$

First we make the assumption that $\zeta$ is constant throughout the whole inner region: We can then write

$$\frac{\partial}{\partial x}\left(f_1-\zeta\ W\right)+\frac{\partial}{\partial y}\left(f_2+\zeta\ \varphi\right)=0$$

$$\frac{\partial}{\partial y}\left(f_1-\zeta\ W\right)-\frac{\partial}{\partial x}\left(f_2+\zeta\ \varphi\right)=0$$

These equations are satisfied if we put

$$f_1-\zeta\ W=\text{constant};\ f_2+\zeta\ \varphi=\text{constant}.$$

By considering equation (12) there follows from the last equation especially

$$k\ W+(\lambda+\zeta)\ \varphi=\text{Constant}$$

$$W=-\frac{\lambda+\zeta}{k}\varphi+\text{Constant}.$$

From the first of these equations we also obtain,

$$k\ \Delta\ W+(\lambda+\zeta)\ \Delta\ \varphi=0$$

or

$$k\ \zeta=c\ (\lambda+\zeta)$$

whence

$$\zeta=\frac{\lambda c}{k-c}\ \text{and}\ W=-\frac{\lambda}{k-c}\cdot\varphi+\text{Constant}$$

But in general the values of $W$ thus found merge continuously into each other at the borders of the two regions quite as little as do their differential quotients. Hence it follows that the component velocities also, and therefore both the velocity and also its direction, suffer sudden changes of finite magnitude at the boundaries of the two regions. We have therefore found only one special solution, and not one that obtains in general. This special solution is that which Guldberg and Mohn have used in the special case of a circular boundary for the inner region. Corresponding to it they find that in the outer region the di-

rection of the wind makes an angle $\varepsilon$ with the radial gradient such that $\tan\varepsilon=\frac{\lambda}{k}$, whereas in the inner region the corresponding angle $\varepsilon'$ is given by the equation $\tan\varepsilon'=\frac{\lambda}{k-c}$.

Still less allowable are the consequences that follow when we imagine the inner region bounded by some other curve such as an ellipse. In this case by utilizing the special solution it results that at special portions of the boundary more air flows inward from without than flows away, but at other special portions of the boundary the relation is reversed. One can easily persuade oneself of this by using the known value of the logarithmic potential of an ellipse.* When therefore $W$ can be considered as the logarithmic potential of a stratum of the inner region still it is not to be considered as constant. Its value is to be specially determined for each given region. This computation will now be executed for the case of a circular region.

## VI. CYCLONE WITH A CIRCULAR INNER REGION.

Let the region of ascending air currents be bounded by a circle of the radius $R$. Let the center of the circle be the origin of the system of co-ordinates. We put

$$r^2=x^2+y^2.$$

First the velocity potential is easily computed as follows:

For an exterior point

$$\varphi_a=-\frac{c}{2}R\log r \quad . \quad . \quad . \quad . \quad . \quad . \quad (23a)$$

For an interior point

$$\varphi_i=-\frac{c}{4}\left\{R^2(2\log R-1)+r^2\right\} \quad . \quad . \quad . \quad (23b)$$

Furthermore for an exterior point we have

$$W_a=\frac{\lambda c}{2k}R\log r$$

Of the functions $\zeta$, $W_i$ and $P$, which are still to be determined, it can certainly be assumed that they depend upon $r$ only.

If we further consider that

$$\frac{\partial f(r)}{\partial x}=\frac{df(r)}{dr}\cdot\frac{x}{r},$$

then equations (13) can be written;

$$x\frac{df_1}{dr}+y\frac{df_2}{dr}=-\zeta\left(y\frac{d\varphi}{dr}-x\frac{dW}{dr}\right)$$

$$y\frac{df_1}{dr}-x\frac{df_2}{dr}=+\zeta\left(x\frac{d\varphi}{dr}+y\frac{dW}{dr}\right)$$

*Kirchhoff, *Vorlesungen über Mechanik*, 1876, page 217.

If we multiply the first by $x$ the second by $y$ and add we obtain

$$\frac{df_1}{dr}=\zeta\frac{dW}{dr}$$

or if we introduce the value of $f_1$

$$\frac{dP}{dr}=-k\frac{d\varphi}{dr}+(\lambda+\zeta)\frac{dW}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If on the other hand the first of the above equations is multiplied by $y$ the second by $x$ and subtracted there results

$$\frac{df_2}{dr}=-\zeta\frac{d\varphi}{dr}$$

or

$$k\frac{dW}{dr}+(\lambda+\zeta)\frac{d\varphi}{dr}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Since furthermore

$$\zeta=\Delta W=\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)$$

and

$$\frac{d\varphi}{dr}=-\frac{c}{2}r,$$

therefore we have in equation (25) an ordinary differential equation for the determination of $W_i$.

If furthermore we put

$$\frac{2k}{c}=\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

then equation (25) becomes

$$r\frac{d}{dr}\left(r\frac{dW}{dr}\right)+\lambda r^2=\mu\left(r\frac{dW}{dr}\right).$$

This gives the following integral where $A$ is the constant of integration:

$$r\frac{dW}{dr}=A\,.\,r^{\mu}+\frac{\lambda}{\mu-2}\cdot r^2.$$

This may finally be written—

$$\frac{dW}{dr}=A\,.\,r^{\mu-1}+\frac{\lambda}{\mu-2}\cdot r.$$

The constant $A$ is now to be so determined that on the borders of both regions, that is to say for $r = R$, the movements pass continuously from one into the other. Since now

$$u=\frac{x}{r}\cdot\frac{d\varphi}{dr}+\frac{y}{r}\frac{dW}{dr};$$

$$v=\frac{y}{r}\cdot\frac{d\varphi}{dr}-\frac{x}{r}\frac{dW}{dr},$$

therefore at the boundary we must have

$$\frac{d\varphi_a}{dr}=\frac{d\varphi_i}{dr} \qquad \frac{dW_a}{dr}=\frac{dW_i}{dr}.$$

This condition is satisfied for the function $\varphi$. We have still to bring it about that the corresponding equation shall be satisfied by the function $W$. Since

$$\frac{dW_a}{dr}=\frac{\lambda c}{2k}r,$$

therefore for $r=R$ we have

$$\frac{dW_i}{dr}=\frac{\lambda c}{2k}\cdot R.$$

This latter will be the case when we put

$$A=\frac{-2\lambda}{\mu(\mu-2)}\cdot R^{2-\mu}$$

Therefore we have finally

$$\frac{dW_i}{dr}=\frac{\lambda}{\mu-2}r\left\{1-\frac{2}{\mu}\left(\frac{r}{R}\right)^{\mu-2}\right\}$$

or if for abbreviation we put

$$f(r)=1-\frac{2}{\mu}\left(\frac{r}{R}\right)^{\mu-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

we obtain

$$\frac{dW_i}{dr}=\frac{\lambda}{\mu-2}r\cdot f(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The function $f(r)$ can according to equation (26) also be written—

$$f(r)=1-\frac{c}{k}\left(\frac{r}{R}\right)^{2\frac{k-c}{c}}$$

This gives $f(r)=1$ for $r=0$ and $f(r)=\frac{(k-c)}{k}$ for $r=R$.

In accordance with these conditions our results are now as follows:

(*a*) for the outer region

$$\left.\begin{aligned} u &= -\frac{c}{2}\frac{R^2}{r^2}\left\{x - \frac{\lambda}{k}y\right\} \\ \omega &= \frac{c}{2}\frac{R^2}{r}\sqrt{1 + \frac{\lambda^2}{k^2}} \\ v &= -\frac{c}{2}\frac{R^2}{r^2}\left\{y + \frac{\lambda}{k}x\right\} \\ \tan\varepsilon &= \frac{\lambda}{k} \end{aligned}\right\} \quad \ldots\ldots \quad (29)$$

(*b*) for the inner portion

$$\left.\begin{aligned} u &= -\frac{c}{2}\left\{x - \frac{\lambda}{k-c}\cdot y\cdot f(r)\right\} \\ \omega &= r\cdot\frac{c}{2}\sqrt{1 + \left(\frac{\lambda}{k-c}\right)^2\left[f(r)\right]^2} \\ v &= -\frac{c}{2}\left\{y + \frac{\lambda}{k-c}\cdot x\cdot f(r)\right\} \\ \tan\varepsilon &= \frac{\lambda}{k-c}\cdot f(r) \end{aligned}\right\} \quad \ldots\ldots \quad (30)$$

In these equations $\varepsilon$ indicates the angle between the direction of the wind and direction of the gradient, which latter coincides of course with the radius of the circle.

These expressions differ from the solutions given by Guldberg and Mohn (not to speak of some small changes in the notation) by the introduction of the function $f(r)$ in whose place the factor 1 is given by them.

The above-given expressions are subject to one limitation. It is necessary that we have $\mu > z$ or $k > c$, since otherwise for $r = o$ $f(r)$ would become infinitely great, and in the inner region a deviation of the wind from the gradient toward the left would occur instead of toward the right-hand side.

The deviation of the wind direction from the gradient is constant in the outer region, but in the inner region it increases continuously and for $r = o$ it attains the limiting value—

$$\tan\varepsilon = \frac{\lambda}{k-c}.$$

I pass now on to the computation of the pressure. According to equation (17) we have for the outer region—

$$P_a = \text{constant} - k\varphi_a\left(1 + \frac{\lambda^2}{k^2}\right)$$

Consequently

$$P_a = \text{constant} + \frac{kc}{2}\left(1+\frac{\lambda^2}{k^2}\right)R^2 \log r.$$

For the inner region the equation (24) is to be used. According to it we have—

$$\frac{dP_i}{dr} = -k\frac{d\varphi}{dr} + (\lambda+\zeta)\frac{dW}{dr}$$

But according to equation (25) we have—

$$\lambda+\zeta = -k\frac{\frac{dW}{dr}}{\frac{d\varphi}{dr}}$$

Therefore,

$$P_i = \text{const} - k\varphi - k.\int\frac{\left(\frac{dW}{dr}\right)^2}{\frac{d\varphi}{dr}}dr.$$

The arbitrary constant can be considered as determined in that the value of $P$ is supposed to be given for $r=0$. (For the center of the depression we have $r=0$ and $P=\frac{p}{\rho}$.) Let $P_0$ be this value. Then we have—

$$P_i = P_0 + F(r),$$

Where

$$F(r) = \frac{kc}{4}r^2\left\{1+\left(\frac{\lambda}{k-c}\right)^2\left(1+\frac{8}{\mu^2}\cdot\left(\frac{r}{R}\right)^{\mu-2}+\frac{4}{\mu^2(\mu-1)}\left(\frac{r}{R}\right)^{2\mu-4}\right)\right\} \quad . \quad (31)$$

Since $P_a$ and $P_i$ must at the boundary merge continuously into each other, therefore the constant in the expression for $P_a$ is to be determined in accordance with this condition, and we have—

$$P_a = P_0 + F(R) + \frac{kc}{2}\left(1+\frac{\lambda^2}{k^2}\right)R^2 \log\frac{r}{R} \quad . \quad . \quad . \quad (32)$$

From equation 9 we obtain the expression for the pressure—

$$\frac{p}{\rho} = P - \tfrac{1}{2}\omega^2$$

If we designate by $p_0$, the pressure at the center of the depression, where $\omega=0$, then in the inner region we have—

$$\frac{p-p_0}{\rho} = F(r) - \tfrac{1}{2}\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

but in the outer region—

$$\frac{p-p_0}{\rho} = F(R) + \frac{kc}{2}\left(1+\frac{\lambda^2}{k^2}\right)R^2 \log\frac{r}{R} - \tfrac{1}{2}\omega^2 \quad . \quad . \quad . \quad . \quad . \quad (34)$$

## VII. NUMERICAL EXAMPLE FOR A CYCLONE: NOTE ON ANTI-CYCLONES.

In order to show the applicability of the formulæ obtained in the last section to cyclones as they actually occur in nature, I have executed the following computation of a numerical example:

In this computation I have assumed

$$\lambda = 0.00012$$

This value corresponds to an average latitude of 55.5°. For $k$ I have assumed the same value, whereby the value obtained for the influence of friction is rather large.

For the complete determination of the system of winds the constant $c$ of the ascending current of air and the dimensions of the inner region must also be known. We can obtain this in various ways. We can assume as given, a definite difference in pressure between the center and a circle of known radius; or on the other hand, we can assume that the velocity of the wind is known at a certain distance from the center. I have chosen the last assumption.

The wind system may therefore be characterized by the assumption that at a distance of 1000 kilometres from the center the wind velocity shall be 10 metres per second.

According to equation (29) when we put $\lambda = k$ we have

$$\omega = \frac{c\ R^2}{\sqrt{2}} \cdot \frac{1}{r}$$

If in this we put $\omega = 10$ metres and $r = 1000000$ metres we then have $c\ R^2 = 10000000\ \sqrt{2}$. Since furthermore $c < k$, therefore the same equation shows that we must have $R > 343.3$ kilometres.

In the selection of appropriate values of $c$ and $R$, another circumstance is to be considered. The discussion of the formulæ (30) for the velocity $\omega$ shows that under the assumption here made of $\lambda = k$, the maximum velocity of the wind occurs at the boundary of the two regions. The smaller the inner region is chosen, by so much larger results the maximum velocity $\omega_R$. In the following table some coördinate values $c$, $\mu$, $R$, and $\omega_R$ are given.

TABLE I.

| $c$ | $\mu$ | $R$ | $\omega_R$ |
|---|---|---|---|
| | | *Kilometres.* | *Metres per sec.* |
| $\frac{4}{5}k$ | $\frac{5}{2}$ | 383.8 | 26.06 |
| $\frac{2}{3}k$ | 3 | 420.4 | 23.78 |
| $\frac{1}{2}k$ | 4 | 485.5 | 20.60 |
| $\frac{1}{3}k$ | 6 | 594.6 | 16.82 |

I have also executed the further complete computation for the first case where $c=\frac{4}{5}k$; the results of this work are given in Table 2. In this computation the equations (29) and (30) were used for the determination of the velocities $\omega$ and the deviations $\varepsilon$ of the direction of the wind from the radial gradient. Furthermore, the differences of pressure $(p-p_0)$ with respect to that at the center, in the circles of radius $r$, were computed according to equations (31), (32), (33) and (34). These latter are, however, converted from the units ordinarily used in hydrodynamics into differences of barometric pressure $(b-b_0)$. This latter is easily done if we recall that for $b=760$ millimetres the ratio $\frac{p}{\rho}$ is equal to the square of the Newtonian velocity of sound; therefore we have the proportion

$$(b-b_0) : 760=\frac{1}{\rho}(p-p_0); (279.9)^2$$

The gradients $\gamma$ are in our present case the differences of barometric pressure for a horizontal distance of 100 kilometres.

TABLE II.

| $r$ | $\omega$ | $\epsilon$ | $(b-b_0)$ | $\gamma$ |
|---|---|---|---|---|
| *Kilometres.* | *Metres per sec.* | ° ′ | *Millimetres.* | *Millimetres.* |
| 0 | 0 | 78 41 | 0 | |
| | | | | 2.37 |
| 100 | 14.99 | 71 19 | 2.37 | |
| | | | | 4.64 |
| 200 | 22.44 | 64 40 | 7.01 | |
| | | | | 5.03 |
| 300 | 25.53 | 55 39 | 12.04 | |
| 383.8 | 26.06 | 45 00 | 15.88 | 4.78 |
| 400 | 25.00 | 45 00 | 16.82 | |
| | | | | 4.76 |
| 500 | 20.00 | 45 00 | 21.58 | |
| | | | | 3.60 |
| 600 | 16.67 | 45 00 | 25.18 | |
| | | | | 2.66 |
| 800 | 12.50 | 45 00 | 30.50 | |
| | | | | 1.95 |
| 1,000 | 10.00 | 45 00 | 34.45 | |

From this table we see that the cyclone includes a broad storm region from $r=200$ to $r=500$ kilometres, of which a portion is in the inner region and another portion in the outer region. Of course the gradients are greatest in the inner region; therefore there the isobars are most crowded together.

From those values of the constant $c$ that are any way possible, it follows that the velocity of the ascending current of air is extraordinarily small; for the present example $c$ equals 0.000096. If we assume that the formula $w=cz$ holds good to an altitude of 1,000 metres, then the vertical velocity would at that height first attain the value of about 0.1 metre per second.

Hitherto the discussion has exclusively dealt with regions of ascend-

ing currents of air and the cyclones arising therefrom. It would be easy in an entirely similar way to develop the theory of descending currents of air and the anti-cyclones resulting therefrom, and here also, as an example, to assume an inner region bounded circularly. Before the actual execution of the exact computation I had believed that this was simply a case of the change of the sign of the constant $c$. But in this operation we stumble upon a peculiar difficulty. The function $f(r)=1-\frac{2}{\mu}\left(\frac{r}{R}\right)^{\mu-2}$ $\left(\text{wherein } \mu=\frac{2k}{c}\right)$ which enters into the expression for the component velocities in the inner region becomes infinitely great for negative values of $c$ and $\mu$ and for $r=o$. The same is true of the function $F(r)$ entering into the expression for the pressure. Hence it follows that the formula just given can not be applied to anti-cyclones with a reversed sign of $c$.

Therefore minima and maxima of pressure show a characteristic difference in their theoretical treatment. But this, as I believe, corresponds also to the real conditions of the true phenomena. Depressions are ordinarily confined to limited areas, but are of considerable intensity, while on the other hand the maxima of pressure extend with slight intensity over broad areas.

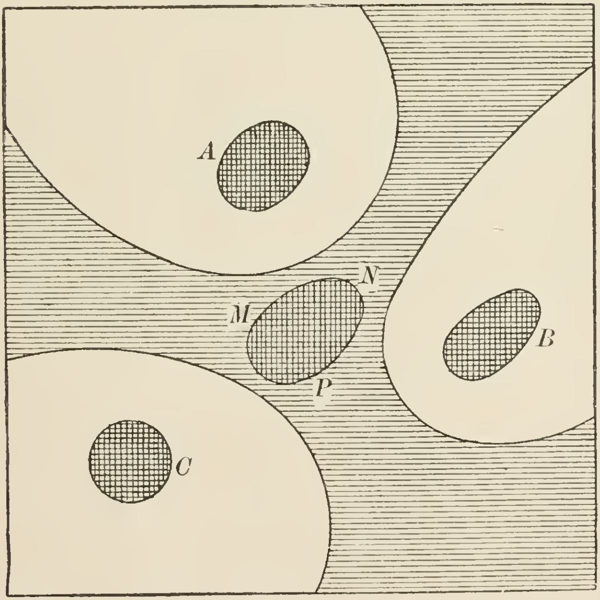



FIG 27.

Moreover, both phenomena stand in close connection, such that one can consider the ascending currents of air as the cause of the descending currents. Hence to a complete cyclone there belong an inner region with ascending air current, a zone surrounding it of purely

horizontal movement, and at a greater distance from the center a ring-shaped region of descending currents.

If we assume that the boundaries of the three regions consist of concentric circles, it would not be difficult to compute the wind system for the whole region by the help of the potential theory as above employed. In this case, where we have to do with an annular region with a descending current of air, the use of the function $f(r)$, even with a negative sign before the $\mu$, is allowable, and can be adopted in order to produce the necessary continuity of motion at the boundary of the two annular regions. If there are several regions of depression with ascending currents of air, as at $A$, $B$, $C$, fig. 27, then each of them is immediately surrounded by a zone of purely horizontal movement, which is bordered by an outside annular zone of descending movement. I have in the figure (27) distinguished the region of ascending and descending current by double and single shading. In the region where the different ring systems of ascending air currents merge into each other there will lie a region of highest pressure with anticyclonal movement of the air somewhat as within the isobar $M$, $N$, $P$. However, the characteristic difference between ascending and descending currents of air always consists in this, that the former consist of definite, simply connected areas; the latter, on the other hand, of a network of several complexly connected regions.

---

HALLE A. S., *June*, 1882.

*P. S.*—After sending the above treatise to the editor of the *Annalen*, I found in the May number of the *Zeitschrift* of the Austrian Association for Meteorology (vol. XVII, pp. 161–175) a review by Dr. A. Sprung of the second part of the collected memoirs by W. Ferrel, under the title of "Meteorological Researches."

From this I perceive that the views expressed by me as to regions with high pressure had been already expressed by Ferrel. Therefore, although my point of view is no longer new, still I rejoice to see that it is shared by a prominent meteorologist.

XI.

# ON THE GULDBERG-MOHN THEORY OF HORIZONTAL ATMOSPHERIC CURRENTS.*

By Prof. Dr. A. OBERBECK, *of the University of Halle.*

Starting from the generally known results of recent meteorological observations in so far as these relate to the distribution of pressure and the direction and force of the wind, the author states that one of the most important problems of the mathematical theory of the motion of fluids is to explain quantitatively the connection of the above-named phenomena. The recently published investigations of Guldberg and Mohn (*Etudes sur les mouvements de l'atmosphère.* Christiania, 1876 and 1880) are to be considered as a specially successful attempt in this direction. It must be of interest also for the larger number of geographers to know the most important results to which the Norwegian scientists have attained.

In order to understand the horizontal movements of the atmosphere it is important for a moment to consider their causes. As such we consider the differences of pressure at the surface of the earth as observed with the barometer. But whence do these arise? This question has been answered a long time since. It is heat which is to be considered as the prime cause of the disturbance of equilibrium in the atmosphere. Because of the slight conductivity of the air the process of warming can progress only slowly from below upwards, so that as is well known the temperature of the air steadily diminishes as we ascend. The heated air expands. The pressure becomes less. If the heating takes place uniformly over a large area there will be at first no reason for horizontal currents. But vertical currents can certainly be brought about by this means. If we imagine a circumscribed mass of air transported into a higher region without any increase or diminution of its heat its temperature will sink because it has expanded itself proportionately to the diminished pressure. If its temperature is then equal to that prevailing in the upper stratum it will remain in equilibrium at this altitude as well as below. The atmosphere in this case exists in a state of indifferent equilibrium. If its temperature is lower the

* Translated from the *Verhandlungen des Zweiten Deutschen Geographentages.* Halle, April, 1882.

mass of air will again sink down; in the reverse case it will rise higher. The air in these cases is then in stable or unstable equilibrium respectively. In the latter case any vertical movement initiated by some accidental disturbance will not again disappear, but rapidly assume increasing dimensions. The current will also continue uniform for a long time.

This is the explanation first given by the mathematician Reye,* of Strasburg, of the ascending air currents in the whirlwinds of the tropics.

The winds of our (temperate) zone also presuppose such ascending currents whose origin must have been quite similar. The ascending current is in general restricted to a definite region that we can designate as the base. Since the ascending current consists of warmer air, therefore above its base the pressure sinks. A barometric depression is inaugurated there. The pressure increases from this region outward in all directions. The isobars therefore surround the region of ascending atmospheric currents in closed curves. At greater heights the upper cooled air flows away to one side and in other regions gives occasion to descending currents of air. At the earth's surface itself, the air flows towards the depression; its influence thus extends over an area much greater than that of the base. If we neglect the curvature of the earth's surface we find over this larger area only simple horizontal movements. Mathematical computations should now reveal to us the nature of such horizontal movements. To this end all the causes of motion, or the forces that come into consideration, are first to be collected.

The differences of pressure have already been several times spoken of. We take as the measure of these differences, the gradient which gives for any point the direction and amount of the greatest change in pressure. In horizontal movements the effect of gravity can be omitted.

On the other hand attention must be given to the rotation of the earth on its axis, since we are only interested in the paths of the winds on the rotating earth. This influence can be taken account of if we imagine at every point of the mass of air a force applied which is perpendicular to the momentary direction of motion and is equal to the product of the double angular velocity of the earth by the sine of the latitude and by the velocity of the point. In the Northern Hemisphere this influence causes a continuous departure of the path towards the right-hand side. Since the movement takes place directly on the earth's surface the direct influence of that surface, namely the friction, remains to be considered. Its influence diminishes with the distance from the earth's surface. Furthermore it depends on the nature of the earth's surface, whether sea or land, plains or wooded mountains. For this computation Guldberg and Mohn have made a convenient assumption in that they introduce the friction as a force which opposes the move-

*[This explanation is of course much older than Reye (1864), who was preceded by Espy and Henry in the United States and by Wm. Thomson in Great Britain. *C. A.*]

ment and is equal to the product of a given factor and the velocity. This factor can have different values according to the nature of the earth's surface [and will be called the friction constant].

All these forces are to be introduced into the general equations of motion of the air. If however one desires solutions of these general equations for special cases there is still needed a series of assumptions.

Let there be only one single vertical current of air present. The totality of all the atmospheric movements depending upon this one vertical current is called a wind-system. If the strength of the ascending current is variable or if the base itself changes its place, then the wind-system is variable. In the first case the system stands still, in the second case it is movable.

If on the other hand the ascending current of air retains its strength and location without change, or, which is the same, if the isobars for a long time retain their position, then the wind system is invariable.

It is evident that the last case is by far the most simple. We will therefore begin with its consideration.

In order to execute the calculation the location of the isobars must be known. Even in this respect also in a preliminary way, one must limit himself at first by simple assumptions. Let the isobars be either parallel straight lines or concentric circles.

In the first case the computation leads to the following simple results:

(1) The parallel isobars are equally distant from each other. The gradient is therefore everywhere of equal magnitude.

(2) The paths of the winds consist of parallel straight lines. The strength of the wind has everywhere the same value.

(3) The direction of the wind forms an angle with the gradient whose tangent is equal to the quotient of the factor arising from the velocity of the earth's rotation divided by the friction constant.

The deviation of the wind from the gradient is therefore greater in proportion as friction is smaller. If the earth's surface were perfectly smooth the wind would blow in the direction of the isobars.

This result, following directly from the computation and at first surprising, finds its confirmation in a variety of observations. For example, in England we observe a deviation of 61° for land winds, but of 77° for sea breezes. From this it follows that the friction on the land is more than twice as great as on the sea.

Conditions of pressure like those here considered frequently occur. In the regions of the trade winds and monsoons they ordinarily prevail either during the whole or about the half of the year.

The circular isobars to the consideration of which we now pass produce systems of wind that can be considered as the simplest types of cyclones and anti-cyclones according as the pressure in the interior is a minimum or maximum. We confine ourselves here to the consideration of cyclones.

As already remarked cyclones are not conceivable without an ascend-

ing current of air, whose area in our case is defined by a circle. Outside of this circle horizontal movements prevail exclusively; inside of it there is also the vertical movement to be considered. Therefore the computations for the outer and inner regions are different. In this way we obtain the following results:

(1) The pressure increases from all sides outward from the center; the gradient increases also from the center out to the limit of the inner region; thence outward it diminishes and at a great distance becomes inappreciable.

(2) The wind-paths in both regions are curved lines, logarithmic spirals, which cut the isobars everywhere at the same angle or make everywhere the same angle with the radial gradient. Therefore the movement of the air can be considered as consisting of a current toward the center and a rotation around the center, the latter in direction opposite to the hands of a watch. This departure from the gradient is of different magnitudes in the outer and inner regions. For the former the departure has the same value as for straight-line isobars, that is to say, it depends alone upon the rotation of the earth and the friction. For the inner region the departure is greater, and depends besides upon the intensity of the ascending current of air. If both regions were separated from each other by a geometrical cylindrical surface then the wind-paths in these would not continuously merge into each other, but would form an angle with each other. This of course can never occur in nature. We must therefore assume a transition region in which the wind is continuously diverted from one into the other direction. At any rate accurate and comparative observation of the wind direction in the inner and outer region of a cyclone would be of great interest. From these one could draw a conclusion as to the limitation of the ascending current of air. This limit is moreover also notable because at it the winds reach their greatest force.

There are no other arrangements that have been discussed theoretically as yet except the straight line and the circular and nearly circular forms of the isobars.

We have as yet only spoken of the invariable systems of wind. In fact however their duration is relatively short. No sooner is a depression formed than it fills up. Furthermore the central region of depression generally does not remain long in the same place but wanders often with great velocity, drawing the whole system of winds with it. We must look to the density of the horizontal current flowing in towards the ascending current of air as the cause of these changes. The system of winds remains unchanged only when, as has hitherto been silently assumed, the temperature and density of the horizontal and vertical currents are alike. If the inflowing air is warmer the depression increases in depth; in the opposite case it becomes shallower.

Finally, if the inflowing air is not of the same temperature on all sides, but has on the one side higher and on the other side lower

temperature than the ascending air, then it will on the one side be strengthened and its area increased, on the other side enfeebled and its area diminished. The consequence of this is that the current of air or the region of depression moves along; the cyclone progresses. Since in the cyclones of our north temperate zone the air entering on the east side comes from more southern—therefore in general—warmer regions, while the air entering on the west side comes from the north and is generally colder, therefore the cyclone progresses from west to east or from southwest to northeast. This is in fact the path of most cyclones in northern Europe. For a moving cyclone the isobaric curves must have a different shape than for one that is stationary; therefore one can inversely from the shape of the isobars infer the direction of motion. If the region of ascending air has a circular form the computation can be rigorously executed. Without going into the details of this interesting problem in this place I will only remark that the isobars consist in closed curves similar to an ellipse. There is one direction from the center outward in which the isobars are most crowded together, while in the opposite direction they are furthest apart. The movement of the cyclone is in a direction at right angles to this line. With the solution of this problem we now stand about at the limits of what analysis has thus far accomplished. Still there is hope that it will make further progress so far as concerns the relation between the pressure and the motion of the air at the earth's surface.

## XII.

## ON THE PHENOMENA OF MOTION IN THE ATMOSPHERE.*

### (*FIRST COMMUNICATION.*)

By Prof. A. OBERBECK, *of the University of Greifswald, Germany.*

### I.

The meteorological observations of the last ten years have given a series of notable laws that principally relate to the connection between the currents of air and the pressure of the air in the neighborhood of the earth's surface.

Of course one can only hope to obtain a complete insight into the complicated mechanism of the motion of the air when one understands more accurately the condition of the atmosphere in its higher strata. But difficulties that are perhaps never to be overcome oppose the observation of these strata. On the other hand, the completion of this and many other gaps in the theory of the motion of the air is certainly to be expected from a comprehensive mechanics of the atmosphere. The Treatise on Meteorology, by A. Sprung, Hamburg, 1885, gives a summary of what has hitherto been accomplished in this field, from which summary it is seen that only special individual problems have found a satisfactory solution.

The principal features of a rational mechanics of the atmosphere are given in the memoir by W. Siemens, "The conservation of energy in the earth's atmosphere."† It appears to me worth while to follow out mathematically the questions there treated of and to develop a theory of the motions of the air as general as possible. The results thus far attained by me, are collected in this present memoir.

On account of the magnitude and difficulty of the problem to be solved, I have at first confined myself to the determination of the currents of the air. A corresponding investigation of the distribution of pressure will follow hereafter. Moreover the phenomena of motion

* Read before the Royal Prussian Academy of Sciences, at Berlin, March 15, 1888. Translated from the *Sitzungsberichte Königl. Preus. Akad. der Wissenschaften.* 1888, pp. 383–395.

† See *Berlin Sitzungsberichte*, 1886, pp. 261–275.

will here be considered as "steady motion." On the other hand I have labored so to arrange the calculation that it can be applied to any condition of the atmosphere and to the general currents between the poles and the equator, or the atmospheric circulation, as well as also to individual cyclones or anticyclones.

In order to test the applicability of the formula thus obtained, the first of the problems just mentioned is completely solved.

I begin with an enumeration of the factors upon which the movement of the atmosphere depends, and with a description of the manner in which I have introduced these into the calculation.

## II

(1) Since the ultimate cause of the motion of the air is to be sought in the effect of gravity and in the differences of temperature in the atmosphere, therefore the attraction of the earth must enter into the equations of motion as the moving force. But it is entirely sufficient here to consider the earth as a homogeneous sphere.

(2) The temperature of the atmosphere is to be considered as a function of the locality, but entirely independent of the time. The last condition is necessary if one confines himself to steady motions. For the temperature $T$, the analytical condition

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

must be satisfied.

This equation, as is well known, follows from the assumption that the heat is distributed through the medium in question according to the laws of the conduction of heat. Although I am by no means of the opinion that the conduction of heat principally determines the flow of heat from the earth's surface through the atmosphere into the planetary space, still it is very probable that the totality of all the phenomena here coming into consideration (conduction, radiation from the earth's surface with partial absorption in the atmosphere, vertical convection currents, etc.) will bring about a distribution of temperature analogous to that due to the conduction of heat.

(3) According to the rules of mechanics, the influence of the rotation of the earth can be expressed by a deflecting force, so that after its introduction the earth can be considered as at rest.

(4) Friction is furthermore to be considered, since without it the atmospheric currents under the continuous influence of accelerating forces would attain to indefinitely great velocities. In my opinion, the attempts made hitherto to give a correct theory of the motions of the air, especially one that can be developed analytically, have failed because of the insufficient or incorrect introduction of friction. I have adhered to the simplest assumption, namely, that the same law of friction holds good for atmospheric currents that has also been shown

to be correct in the motion of liquids.* But I would not hereby assert that the same numerical coefficient is to be used as is given by the laboratory experiments on the internal friction of the air made under the exclusion of all attendant disturbing circumstances. More likely is it that along with the greater horizontal currents there will arise small vertical currents of a local nature which will increase the friction. The air can either be held fast at the earth's surface or glide with more or less resistance. This fact, as is well known, is expressed in the boundary equations of condition by a number, the coefficient of slip, whose value may lie between zero and infinity.

(5.) The density of the air must be considered as dependent upon the temperature, since the effective cause of the currents results from this. But I have not objected to use, as the equation of continuity, that simpler expression that obtains for incompressible liquids. The error introduced hereby can be eliminated if, at places where the density is less than the average, one increases to a corresponding extent the velocity found for that locality, but considers the velocity as diminished at locations where the density exceeds the average.

(6) A hydro-dynamic problem is only perfectly definite when the fluid occupies a definite space, and its behavior is known for all limiting boundary surfaces. I have therefore assumed that the atmosphere is bounded both by the earth's surface and by a second spherical surface concentric therewith. The distance of the two spherical surfaces, which I will briefly designate as the height of the atmosphere, can remain undetermined. But this is quite small in comparison with the earth's radius. The above assumption just made however, only expresses the idea that for a given altitude above the earth's surface the radial or vertical currents are very small, or rather that when they are present they exert an inappreciably small influence on the remaining motions. This is certainly the case, since at very large altitudes the density is very small. Since moreover it is assumed that the air can glide without resistance on the upper spherical surface, therefore in my opinion no limitation of the motions of the atmosphere, contradictory to the real phenomena, results from the introduction of such an upper boundary surface.

## III.

The following notation will be used for the principal equations of the problem. The position of a point in the atmosphere is determined by the rectangular coördinates $x$, $y$, $z$. The center of the earth is the origin of coördinates and the earth's axis in the direction of the North Pole is the positive axis of $z$. The positive directions of the two other axes are to be so chosen that the axis of $y$ as seen from the North Pole must be turned through an angle of 90° in the direction of the motion of the hands of a watch in order to be made to coincide with the axis of $x$.

*[The term friction as here used therefore includes viscosity and slip, but excludes the resistance due to wave motion and to vortex motion and all the resistances implied in turbulent flow of fluids.—*C. A.*]

Let there be furthermore—

$u, v, w$, the components of velocity;

$p$, the pressure;

$\mu$, the density;

$k$, the coefficient of friction;

$G$, the acceleration of gravity;

$R$, the radius of the earth;

$r$, the distance of any point from the center of the earth;

$\varepsilon$, the angular velocity of the earth.

Then we have—

$$\left.\begin{aligned}
\frac{du}{dt} &= GR^2\frac{\partial\frac{1}{r}}{\partial x} - \frac{1}{\mu}\frac{\partial p}{\partial x} + \frac{k}{\mu}\Delta u + 2\varepsilon v, \\
\frac{dv}{dt} &= GR^2\frac{\partial\frac{1}{r}}{\partial y} - \frac{1}{\mu}\frac{\partial p}{\partial y} + \frac{k}{\mu}\Delta v - 2\varepsilon u, \\
\frac{dw}{dt} &= GR^2\frac{\partial\frac{1}{r}}{\partial z} - \frac{1}{\mu}\frac{\partial p}{\partial z} + \frac{k}{\mu}\Delta w, \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \quad \dots\dots\dots (1)$$

Since according to the law of Mariotte and Gay-Lussac

$$\frac{p}{\mu} = \frac{p_0}{\mu_0}(1 + \alpha T)$$

we may put

$$\frac{1}{\mu}\cdot\frac{\partial p}{\partial x} = \frac{p_0}{\mu_0}(1 + \alpha T)\frac{\partial \log p}{\partial x}.$$

The zero point of temperature is arbitrary. It is most appropriate to assume for it the average temperature of the atmosphere.

If $c$ is the Newtonian value of the velocity of sound, then we have

$$\frac{p_0}{\mu_0} = c^2$$

After the introduction of these expressions into the above principal equations, imagine the latter divided throughout by $1 + \alpha T$. Excepting in that member in which the gravity occurs, one can omit from consideration the influence of the factor $\frac{1}{1 + \alpha T}$. In the term just mentioned one can, as a first approximation, put $(1 - \alpha T)$ for the value of this factor. Furthermore let

$$\frac{k}{\mu} = \varkappa$$

The first of the equations of motion now becomes

$$\frac{du}{dt} = (1-\alpha T)\, GR^2 \frac{\partial \frac{1}{r}}{\partial x} - c^2 \frac{\partial \log p}{\partial x} + \varkappa \Delta u + 2\varepsilon v.$$

If the temperature of the atmosphere depended only on the altitude above the earth's surface and were therefore only a function of $r$, then would these equations be fulfilled by putting $u$, $v$, $w$ respectively $=0$; the atmosphere would then be in equilibrium. Therefore put

$$T = T_0 + T_1$$

wherein $T_0$ is a function of $r$ only, but $T_1$ is also a function of the longitude and latitude; therefore

$$T_0 \frac{\partial \frac{1}{r}}{\partial x} = -\frac{\partial}{\partial x} \int \frac{T_0}{r^2}\, dr$$

$$T_1 \frac{\partial \frac{1}{r}}{\partial x} = \frac{\partial \frac{T_1}{r}}{\partial x} - \frac{1}{r} \frac{\partial T_1}{\partial x}$$

Finally one may put

$$p = p_1 \cdot (1 + \nu).$$

The quantity $\nu$ in this latter equation expresses those changes of pressure that are caused by the phenomena of motion. Since $\nu$ is small in comparison with unity, therefore instead of $\log (1 + \nu)$ the quantity $\nu$ itself can be substituted. By this means the first principal equation becomes

$$\frac{du}{dt} = GR^2 \frac{\partial}{\partial x} \left\{ \frac{1 - \alpha T_1}{r} + \alpha \int \frac{T_0}{r^2}\, dr \right\} - c^2 \frac{\partial \log p_1}{\partial x} - c^2 \frac{\partial \nu}{\partial x} + \varkappa \Delta u + 2\varepsilon v$$

After transforming the two other principal equations in the same manner we can put

$$c^2 \log p_1 = \text{constant} + GR^2 \left\{ \frac{1 - \alpha T_1}{r} + \alpha \int \frac{T_0}{r^2}\, dr \right\} \quad . \quad . \quad (2)$$

This equation gives the diminution of pressure at larger altitudes above the earth's surface, and can for smaller differences of altitude easily be transformed into the ordinary equation of barometric hypsometry.

The following system of equations relating to the phenomena of motion proper now remains:

$$\left.\begin{aligned}
\frac{du}{dt} &= \frac{\alpha G R^2}{r} \cdot \frac{\partial T_1}{\partial x} - c^2 \frac{\partial \nu}{\partial x} + \varkappa \Delta u + 2\varepsilon v, \\
\frac{dv}{dt} &= \frac{\alpha G R^2}{r} \cdot \frac{\partial T_1}{\partial y} - c^2 \frac{\partial \nu}{\partial y} + \varkappa \Delta v - 2\varepsilon u, \\
\frac{dw}{dt} &= \frac{\alpha G R^2}{r} \cdot \frac{\partial T_1}{\partial z} - c^2 \frac{\partial \nu}{\partial z} + \varkappa \Delta w, \\
\frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \quad \ldots \ldots \ldots (3)$$

One can now compute first those components of the current that depend only on temperature differences; after that those that are brought about by the rotation of the earth. If we put $u=u_1+u_2$; $v=v_1+v_2$; $w=w_1+w_2$; $\nu=\nu_1+\nu_2+\nu_3$, then will the following two systems of equations be those that are first to be discussed:

$$c^2 \frac{\partial \nu_1}{\partial x} = \frac{\alpha G R^2}{r} \cdot \frac{\partial T_1}{\partial x} + \varkappa \Delta u_1$$

$$c^2 \frac{\partial \nu_1}{\partial y} = \frac{\alpha G R^2}{r} \cdot \frac{\partial T_1}{\partial y} + \varkappa \Delta v_1$$

$$c^2 \frac{\partial \nu_1}{\partial z} = \frac{\alpha G R^2}{r} \frac{\partial T_1}{\partial z} + \varkappa \Delta w_1$$

and

$$c^2 \frac{\partial \nu_2}{\partial x} = 2\varepsilon v_1 + \varkappa \Delta u_2;$$

$$c^2 \frac{\partial \nu_2}{\partial y} = -2\varepsilon u_1 + \varkappa \Delta v_2;$$

$$c^2 \frac{\partial \nu_2}{\partial z} = \varkappa \Delta w_2.$$

Thus there still remain the following equations which are no longer linear and which will serve principally in the computation of the variations in pressure produced by the motion:

$$c^2 \frac{\partial \nu_3}{\partial x} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 2\varepsilon v_2;$$

$$c^2 \frac{\partial \nu_3}{\partial y} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -2\varepsilon u_2;$$

$$c^2 \frac{\partial \nu_3}{\partial z} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = 0.$$

The first two systems of equations are linear. When therefore $T_1$ consists of a sum of terms we shall obtain corresponding sums for the

component velocities. The solution will be quite simple when $T_1$ is developed into a series of spherical harmonics.

If we put

$$T_1 = \Sigma \left\{ A_n r^n + \frac{A'_n}{r^{n+1}} \right\} p_n$$

and for brevity

$$\beta = \alpha G R^2,$$

and indicate by $Q$ any term of the series with its corresponding constant then the solutions of the first two systems of equations are as follows:

$$\left.\begin{aligned} u_1 &= \frac{\beta}{\varkappa} \left\{ E \frac{\partial Q}{\partial x} + \frac{\partial (QF)}{\partial x} \right\} \\ v_1 &= \frac{\beta}{\varkappa} \left\{ E \frac{\partial Q}{\partial y} + \frac{\partial (QF)}{\partial y} \right\} \\ w_1 &= \frac{\beta}{\varkappa} \left\{ E \frac{\partial Q}{\partial z} + \frac{\partial (QF)}{\partial z} \right\} \\ c^2 \nu_1 &= \beta \{ \Delta (QF) + a Q \} \end{aligned}\right\} \quad \dots \dots \quad (4)$$

In this $E$ and $F$ are functions of $r$ only, and must satisfy the differential equations

$$\left.\begin{aligned} &\left( \frac{d^2 E}{dr^2} + \frac{2}{r} \frac{dE}{dr} \right) \frac{\partial Q}{\partial r} + 2 \frac{dE}{dr} \frac{\partial^2 Q}{\partial r^2} = \frac{\partial Q}{\partial r} \left( -\frac{1}{r} + a \right) \\ &\left( \frac{d^2 F}{dr^2} + \frac{2}{r} \frac{dF}{dr} \right) Q + \frac{\partial Q}{\partial r} \left( 2 \frac{dF}{dr} + \frac{dE}{dr} \right) = 0 \end{aligned}\right\} \quad \dots \quad (5)$$

The constant $a$ must be added in order to obtain the number of constants needed in the consideration of the boundary conditions. The terms depending upon the earth's rotation are

$$\left.\begin{aligned} u_2 &= \frac{2\varepsilon\beta}{\varkappa^2} \left\{ -\left( J \frac{\partial Q}{\partial y} + \frac{\partial (QH)}{\partial y} \right) + \frac{\partial K}{\partial x} \right\} \\ v_2 &= \frac{2\varepsilon\beta}{\varkappa^2} \left\{ +J \frac{\partial Q}{\partial x} + \frac{\partial (QH)}{\partial x} + \frac{\partial K}{\partial y} \right\} \\ w_2 &= \frac{2\varepsilon\beta}{\varkappa^2} \cdot \frac{\partial K}{\partial z} \\ c^2 \nu_2 &= \frac{2\varepsilon\beta}{\varkappa} \Delta K \end{aligned}\right\} \quad \dots \quad (6)$$

Here also $J$ and $H$ are functions of $r$ only, and must satisfy the differential equations

$$\left.\begin{aligned} &\left( \frac{d^2 J}{d r^2} + \frac{2}{r} \frac{dJ}{dr} \right) \frac{\partial Q}{\partial r} + 2 \frac{dJ}{dr} \frac{\partial Q}{\partial r} = \frac{\partial Q}{\partial r} (E - b) \\ &\left( \frac{d^2 H}{dr^2} + \frac{2}{r} \cdot \frac{dH}{dr} \right) Q + 2 \frac{dH}{dr} \cdot \frac{\partial Q}{\partial r} = Q (F + b). \end{aligned}\right\} \quad \dots \quad (7)$$

The constant $b$ must also here be added for the same reason as above given.

The function $K$ is to be computed from the equation

$$\Delta K+\frac{dJ}{dr}\left(\frac{\partial Q}{\partial y}\cdot\frac{x}{r}-\frac{\partial Q}{\partial x}\cdot\frac{y}{r}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From this last equation it follows that the introduction of the function $K$ can be omitted when the temperature of the atmosphere is assumed symmetrical with reference to the earth's axis. In this case $w_2=0$ and the [atmospheric] movement resulting from the rotation of the earth consists exclusively in a movement of rotation depending on the geographical latitude and the altitude above the earth's surface.

In order to present in the ordinary manner the currents of air for a given point in the atmosphere, the following components are to be introduced instead of $u$, $v$, $w$:

$V$, the vertical component computed positively upwards;

$N$ and $O$, the two horizontal components, of which the first indicates movement toward the north, the latter, movement toward the east;

$\theta$, the complement of the geographical latitude of a given place;

$\psi$, the longitude counted from an arbitrary meridian;

then we have

$$\left.\begin{aligned} V&=+(u\cos\psi+v\sin\psi)\sin\theta+w\cos\theta\\ N&=-(u\cos\psi+v\sin\psi)\cos\theta+w\sin\theta\\ O&=-u\sin\psi+v\cos\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

The formulæ (4, 6, and 9) contain the general solution of the problem so far as this is at present intended to be given, assuming the distribution of temperature to be given and that the functions $E$, $F$, $J$, $H$, $K$ are determined in accordance with the boundary conditions.

## IV.

When one attempts to represent the distribution of temperature on the earth's surface by a series of harmonic functions then the most important term is a harmonic function of the second order. Therefore as a first approximation we put

$$T_1=\left(Ar^2+\frac{A'}{r^3}\right)(1-3\cos^2\theta).$$

This function, with a proper determination of the constants, expresses the great contrast in temperature between the equator and the pole. If now one would take into account the variation with the seasons one must next introduce harmonic functions of the first order. The consideration of the various peculiarities of the earth's surface will of course demand further terms that depend on the geographical longitude also.

I have at first limited myself to the computation for the above given distribution of temperature, and put

$$Q = Ar^2\,(1-3\cos^2\theta)$$
$$Q' = \frac{A'}{r^3}\,(1-3\cos^2\theta).$$

The functions $E, F, H, J$ are now to be computed with the help of this $Q$, and the corresponding $E', F', H'$, and $J'$ with the help of this $Q'$. We first obtain the general expressions:

$$V = \frac{\alpha GR^2}{\varkappa}(1-3\cos^2\theta)\left[A\left\{r^2\frac{dF}{dr}+2r\,(F+E)\right\}\right.$$
$$\left.+\frac{A'}{r^4}\left\{r\frac{dF'}{dr}-3\,(E'+F')\right\}\right]$$

$$N = -\frac{\alpha GR^2}{\varkappa}\,6\cos\theta.\sin\theta\left\{Ar(F+E)+\frac{A'}{r^4}(F'+E')\right\}$$

$$O = \frac{\alpha GR^2 2\varepsilon}{\varkappa^2}\sin\theta\left[(1-3\cos^2\theta)\left\{Ar\left(r\frac{dH}{dr}+2\,(H+J)\right)\right.\right.$$
$$\left.\left.+\frac{A}{r^4}\left(r\frac{dH'}{dr}-3\,(H'+J')\right)\right\}+6\cos^2\theta\left\{Ar\,(H+J)+\frac{A'}{r^4}\,(H'J')\right\}\right]$$

The actual computation, having due reference to the boundary conditions, of the functions here introduced, gives results that are difficult to be discussed. But this is simplified when we make use of the circumstance that the atmosphere fills a very thin shell in comparison with the terrestrial sphere, wherefore the distances from the earth's surface are all small in comparison with the earth's radius. If we put

$$r = R\,(1+\sigma)$$

then is $\sigma$ small with respect to unity. If we introduce these quantities in the above given equations and put

$$r\frac{dF}{dr}+2\,(E+F)=Rf(\sigma), \qquad F+E=R\varphi(\sigma);$$

$$r\frac{dF'}{dr}-3\,(E'+F')=Rf'\,(\sigma), \qquad F'+E'=R\varphi'\,(\sigma);$$

$$r\frac{dH}{dr}+2(H+J)=R^3g(\sigma), \qquad H+J=R^3\gamma(\sigma)$$

$$r\frac{dH'}{dr}-3(H'+J')=R^3g'(\sigma), \qquad H'+J'=R\nu^3\gamma'\,(\sigma);$$

then by restricting ourselves to the terms of the lowest order, we can obtain simple expressions for these functions. Primarily we find that the functions $f$ and $f'$, $\varphi$ and $\varphi'$, $g$ and $g'$, $\gamma$ and $\gamma'$ are identical.

Moreover the two constants $A$ and $A'$, which occur in the combination

$$A\ R^2 + \frac{A'}{R^3}$$

can be expressed in terms of the temperatures of the earth's surface at the equator, $T_a$, and at the pole, $T_p$. We have

$$\frac{1}{3}\ (T_a - T_p) = A\ R^2 + \frac{A'}{R^3}$$

Finally we put

$$C = \frac{\alpha\ G\ R^2}{\varkappa}\ \tfrac{1}{3}\ (T_a - T_p)$$

$$D = \frac{\alpha\ G\ R^4}{\varkappa^2}\ 2\varepsilon .\ \tfrac{1}{3}\ (T_a - T_p).$$

The numerical value of these two last constants can not be given, since, as before remarked, the coefficient of friction, $\varkappa$, will not agree with that determined from laboratory experiments. In any case $D$ is considerably larger than $C$, since in $D$ the fourth power of the radius of the earth occurs, but in $C$ only the second power. The components of motion of the atmosphere are, therefore:

$$V = C\ (1 - 3\cos^2\theta) . f(\sigma)$$
$$N = -C .\ 6\cos\theta\sin\theta . \varphi(\sigma)$$
$$O = D\sin\theta \left\{ (1 - 3\ \cos^2\ \theta)\ g\ (\sigma) + 6\cos^2\theta\ \gamma\ (\sigma) \right\}$$

If we take $R.h$ for the altitude of the atmosphere as above defined, then the four functions, $f$, $\varphi$, $g$, $\gamma$, are to be so determined that they satisfy the prescribed boundary conditions for $\sigma = 0$ and $\sigma = h$. I have executed this computation for the most general case, namely, that in which at the upper limit slipping occurs without friction, but at the lower limit sliding with friction. Undoubtedly however the condition of the atmosphere on the earth's surface is much more nearly that of adhesion than that of free slipping, so that I will here communicate only the solutions for this latter case. For this case the motion at the earth's surface is everywhere zero. But for this motion one can easily substitute the motion at a slight altitude, that is to say, for small values of $\sigma$. For the four functions we find the following expressions:

$$f(\sigma) = \frac{\sigma}{8}(h - \sigma)\ (3h\sigma - 2\sigma^2)$$

$$\varphi(\sigma) = \frac{\sigma}{48}\left\{ 6h^2 - 15h\sigma + 8\sigma^2 \right\}$$

$$g\ (\sigma) = \frac{\sigma}{480}\left\{ -9h^5 + 15h^2\sigma^3 - 15h\sigma^4 + 4\sigma^5 \right\}$$

$$\gamma(\sigma) = \frac{\sigma}{960}\left\{ 20h^2\sigma^2 - 25h\sigma^3 + 8\sigma^4 \right\}$$

According to this solution the following gives a picture of the atmospheric circulation, which in its principal points agrees with that of W. Siemens.

(1) *Currents on a spheroid without rotation.*

These currents consist of currents in the meridian, and of vertical movements.

(*a*) The meridional current in the northern hemisphere is southerly below, but northerly above, since the function $\varphi$ changes its sign when $\sigma$ increases from zero to $h$. It attains its largest value at 45°, and disappears at the equator and at the poles.

(*b*) The vertical circulation is zero at the earth's surface and at the upper limit of the atmosphere. From the equator to 35° 16′ north and south latitudes the flow of air is positive—that is to say, ascending—but in higher latitudes it is descending. Its velocity at the poles is twice as great as that at the equator.

By the comparison of the expressions for $f(\sigma)$ and $\varphi(\sigma)$, it appears that the former function contains the fourth powers of the small quantities $h$ and $\sigma$; the latter function contains their third powers. Therefore, the vertical flow is to the horizontal flow, so far as magnitude is concerned, as $h$ is to 1, or as the altitude of the earth's atmosphere is to the radius of the earth. From this we can scarcely assume that we should be successful in the direct observation of the vertical current. The great effect of the vertical current arises from this, that it rises or sinks over a very extensive area.

(2) *Currents in consequence of the rotation of the earth.*

Under the assumption here made as to the distribution of temperature on the earth's surface, these currents consist exclusively of movements along the parallel circles of latitude. As in the case of the two terms in the component $O$, so here we distinguish the two following.

(*a*) The movement depending on the function $g(\sigma)$. Since this function is invariably negative; therefore to begin with at the equator the motion is directed toward the west. It changes its sign at latitude 35° 16′, and then becomes a motion directed toward the east.

(*b*) The second current is zero at the equator; becomes a maximum at 54° 44′, and is exclusively directed toward the east. Both currents disappear at the poles.

The two motions (*a*) and (*b*) differ from each other fundamentally in that $\gamma(\sigma)$ differs from zero first when $\sigma$ has larger values. It is therefore a current that only occurs in the higher strata of the atmosphere. But thereby the function $g$ is of a higher order than $\gamma$ for the small quantities $h$ and $\sigma$. Therefore at great altitudes the current (*b*) must greatly exceed the current (*a*) in velocity.

The components 1*a* and 2*a* combine at the earth's surface to form the regular movement of the air that we designate as the lower trade wind.

On the ocean where this system of winds can freely develop in the manner here assumed, without the influence of continents, their course is in good agreement with the conclusions of theory. Thus, on the northern hemisphere, between 0° and 35° latitude, east and northeast winds prevail; at 35° nearly north or in general only feeble winds; in higher latitudes northwest and west winds.

It results from the preceding that the two currents (1*a*) and (1*b*) are of the same order of magnitude and give moderate winds in the lower strata of atmosphere. Since now the current (2*b*), in comparison with (2*a*) is of a different order of magnitude, therefore the former is by far the most intense of all currents of air, but only in the upper strata of the atmosphere.

In so far as this component combines with the upper current (1*a*), it forms in the tropics the southwest or upper trade wind. In higher latitudes the purely westerly current prevails. So far as is known to me, the observations of the highest clouds which show prevailing west winds agree herewith. That the just-mentioned rotation-currents attain a great velocity has its reason in this that they can circulate around the whole earth without being hindered by the friction of a lower opposite current, as for instance is the case with the meridional currents. I consider it probable (as also W. Siemens has already announced) that in this powerful upper current we have to seek for the principal source of the energy found in the wind system of the lower strata.

# XIII.

## ON THE PHENOMENA OF MOTION IN THE ATMOSPHERE.*

*(SECOND COMMUNICATION.)*

By Prof. A. OBERBECK, *of Greifswald.*

### I.

A comparison of the highest and lowest atmospheric temperatures at the surface of the earth shows permanent differences of 70°C. If the pressure were uniform everywhere these would correspond to differences of density of the air of more than 20 per cent. Since, however, pressure and density mutually influence each other one should therefore expect minima of pressure at places of highest temperature and maxima of pressure at places of low temperature of a corresponding intensity.

Instead of this the average differences of pressure on the earth's surface attain only 6 or 7 per cent., and even the largest rapidly passing barometric variations scarcely exceed 10 per cent. We explain the relatively small value of these differences of pressure by the formation of corresponding currents; a lower current at the earth's surface in the direction of the increasing temperature and an opposite upper current. Still the above-mentioned rule as to the connection between temperature and pressure must be true in general. But this is by no means always the case. While the equatorial zone of highest temperature shows a feeble minimum of pressure there occurs a maximum of pressure between the twentieth and fortieth degree of latitude from which toward either pole, and especially markedly in the southern hemisphere, the atmospheric pressure very decidedly sinks.

It appears to me not to be doubted that we can explain this remarkable phenomenon only by the influence of the rotation of the earth upon the currents of air that originate in temperature differences. In a previous memoir † I have endeavored to carry out an analytical treatment of these phenomena of motion under certain assumptions which

* Read before the Royal Prussian Academy of Sciences at Berlin, November 8, 1888. Translated from the *Sitzungsberichte Königl. Preus. Akad. der Wissenschaften* zu Berlin. 1888, pp. 1129–1138.

† [See the previous number (XII) of this collection of Translations.—*C. A.*]

are there given in detail. In that memoir the pressures were not explained; this is done in the present treatise. I have arrived thus at the result that the distribution of pressure just described finds its explanation completely in the currents of the atmosphere, and that from the observed values of the pressure a conclusion can be drawn as to the intensity of the atmospheric currents.*

II.

In conformity with the notation of my first memoir the temperature of the atmosphere will be expressed by

$$T = T_0 + T_1 .$$

where $T_0$ depends only upon $r$, the distance of the point in question from the center of the earth, while $T_1$ is a function of $r$ and of $\theta$, the polar distance.

Let the pressure at the given point be

$$p=p_0(1+\nu)$$

In this expression $p_0$ also depends only upon $r$, while $\nu$ is a function of $r$ and $\theta$. So far as the observations of atmospheric pressure show, $\nu$ can be considered as a small numerical quantity in comparison with unity. For determining $p_0$ the following equation holds good:

$$c^2 \log p_0 = \text{constant} + GR^2\left(\frac{1}{r}+\alpha\int\frac{T_0}{r^2}dr\right)$$

from which the diminution of pressure as a function of the altitude above the earth's surface can be computed when the law of the diminution of temperature with the altitude, that is to say, the value of $T_0$ as a function of $r$ is known.

Let us further put

$$\nu=\nu_0+\nu_1+\nu_2+\nu_3$$

in which

$$\nu_0=-\frac{GR^2\alpha T_1}{r}$$

while $\nu_1$, $\nu_2$, $\nu_3$ shall indicate the values determined in the previous memoir (pages 180 and 181).

The first two terms of this summation $\nu_0+\nu_1$ give those changes in pressure which result directly from the differences of temperature on the earth's surface; that is to say, without considering the rotation of the earth.

If the temperature diminishes uniformly on both hemispheres from the equator toward the poles; or, in other words, if the temperature

* [Ferrel had published similar conclusions in 1859 but Oberbeck's independent confirmation is none the less valuable.—*C. A.*]

depends only on the geographical latitude (and not also on the longitude), then the motion of the air can only consist in vertical and meridional currents, and which (corresponding to the above given component velocities $u_1$, $v_1$, $w_1$) consist of one lower current toward the equator and of one upper current toward the poles. The distribution of pressure $\nu_0+\nu_1$ existing in connection with this furnishes (by means of the equation (4), page 182 of the previous memoir) the anticipated result that on the surface of the earth the pressure increases from the equator toward the pole, while at a medium altitude the differences of pressure disappear, but that finally, at greater altitudes, the pressure is greatest at the equator and least at the poles.

Since as above remarked, the actual distribution of pressure in nowise agrees with the above, it must be concluded that the influence of the term $\nu_0+\nu_1$ on the pressure can only be slight.

From the previous developments it results that the term $\nu_2$ disappears under the assumption of a uniform distribution of temperature symmetrical with the earth's axis, so that as was already indicated in the first memoir, $\nu_3$ will be the most important term.

## III.

In the computation of this quantity $\nu_3$ the system of equations previously given is to be used, namely:

$$c^2\frac{\partial \nu_3}{\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=2\varepsilon v_2$$

$$c^2\frac{\partial \nu_3}{\partial y}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-2\varepsilon u_2$$

$$c^2\frac{\partial \nu_3}{\partial z}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0$$

Since according to the accordant opinion of meteorologists, as also according to my previous deductions, it is very probable that the intensity of the rotatory currents of the atmosphere materially exceeds that of the meridional currents, therefore I have only introduced into the further computation the rotation currents, whose components are designated by $u_2$ and $v_2$.

Since we have to do with a movement of rotation about the axis of $z$ therefore we can put

$$u_2=-\chi y,\ v_2=+\chi x,\ w_2=0,$$

and these values can also be used for $u$, $v$, and $w$, in the above-given system of equations.

The relative angular velocity $\chi$ is to be deduced from the expression for the easterly component $O$ (see equation (9), page 183). This is a func-

tion of $\theta$ and of $r$ or also of $\sigma$ the altitude above the earth's surface. The first system of equations is therefore transformed into the following:

$$c^2\frac{\partial\nu_3}{\partial x}=(2\varepsilon+\chi)\,\chi x,$$

$$c^2\frac{\partial\nu_3}{\partial y}=(2\varepsilon+\chi)\,\chi y,$$

$$c^2\frac{\partial\nu_3}{\partial z}=0.$$

Since $\chi$ is a function of $r$ and $\theta$, or of $\rho$ and $z$ if we put

$$z=r\cos\theta$$

$$\rho=r\sin\theta;$$

therefore, we can not find one function $\nu_3$ that shall satisfy the three equations. If $\chi$ were independent of $z$ we should find

$$c^2\nu_3=\text{constant}+\int(2\varepsilon+\chi)\,\chi\rho\,d\rho.$$

Since however this is not the case we must therefore conclude that the above-given system of equations still needs a supplement; that therefore a movement of rotation of a fluid to the exclusion of all other movements can only exist when the angular velocity in the direction of the axis of rotation is everywhere the same. If this is not the case then further currents occur perpendicular to the rotary motion. In our case these latter would consist of vertical and meridional movements. Their components may be designated by $u_3$ $v_3$ $w_3$. These are to be introduced into the above system of equations as was done in the corresponding fundamental equations (3) of the first memoir which now become

$$\left.\begin{aligned}c^2\frac{\partial\nu_3}{\partial x}&=(2\varepsilon+\chi)\,\chi x+\varkappa\,\Delta u_3\\ c^2\frac{\partial\nu_3}{\partial y}&=(2\varepsilon+\chi)\,\chi y+\varkappa\Delta v_3\\ c^2\frac{\partial\nu_3}{\partial z}&=\varkappa\Delta w_3\\ \frac{\partial u_3}{\partial x}+\frac{\partial v_3}{\partial y}+\frac{\partial w_3}{\partial z}&=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(2)$$

If the component motions indicated by the subscript 3 that directly depend on the movements subscript 1 are materially less in intensity than the movements of rotation, then in any computation of the pressure their introduction ought not to be omitted. The former memoir gave

a rather complicated value for the angular velocity $\chi$. I have introduced a simplified expression for this in that, while retaining the dependence upon the polar distance $\theta$, as there given, I have temporarily introduced a constant average value instead of the dependence upon the distance above the surface of the earth. According to this, one can put

$$\chi=\chi_1 \cos^2 \theta-\chi_2 \tag{3}$$

or with a slight difference

$$\chi=\frac{1}{R^2}\left\{ \chi_1 z^2-\chi_2 r^2 \right\} \tag{4}$$

In these equations $\chi_1$ and $\chi_2$ are considered as constants. Therefore, as before found, the movement of rotation of the air in higher latitudes is positive, that is to say, has the same sign as the axial rotation of the earth. For a specific latitude the average value is 0, and at the equator the movement has the opposite sign.

Further computation shows that the relative angular velocity $\chi$ is small in comparison with that of the earth $\varepsilon$, so that the simpler equations to be solved are as follows:

$$\left.\begin{aligned} c^2\frac{\partial \nu_3}{\partial x} &= 2\varepsilon\chi x + \varkappa \Delta u_3 \\ c^2\frac{\partial \nu_3}{\partial y} &= 2\varepsilon\chi y + \varkappa \Delta v_3 \\ c^2\frac{\partial \nu_3}{\partial z} &= \qquad\qquad \varkappa \Delta w_3 \\ \frac{\partial u_3}{\partial x}+\frac{\partial v_3}{\partial y}+\frac{\partial w_3}{\partial z} &= 0 \end{aligned}\right\} \quad \cdots\cdots \tag{5}$$

In solving these we first determine a function $\mathfrak{F}$ that is of such form as to satisfy the conditions $\frac{\partial \mathfrak{F}}{\partial x}=2\varepsilon\chi x, \frac{\partial \mathfrak{F}}{\partial y}=2\varepsilon\chi y.$

These conditions give

$$\mathfrak{F}=\frac{\varepsilon r^2}{R^2}\left\{ \chi_1 z^2 - \frac{\chi_2}{2} r^2 \right\} \tag{6}$$

Furthermore we put

$$u_3=\frac{\partial L}{\partial x},\ v_3=\frac{\partial L}{\partial y},\ w_3=\frac{\partial L}{\partial z}+M \quad \cdots\cdots \tag{7}$$

where $L$ and $M$ are two new functions of $x$, $y$, and $z$, we can then write the system of equations as follows:

$$c^2\frac{\partial \nu_3}{\partial x}=\frac{\partial \mathfrak{F}}{\partial x}+\varkappa\frac{\partial}{\partial x}(\Delta L)$$

$$c^2\frac{\partial \nu_3}{\partial y}=\frac{\partial \mathfrak{F}}{\partial y}+\varkappa\frac{\partial}{\partial y}(\Delta L)$$

$$c^2\frac{\partial \nu_3}{\partial z}=\frac{\partial \mathfrak{F}}{\partial z}+\varkappa\frac{\partial}{\partial z}(\Delta L)-\frac{\partial \mathfrak{F}}{\partial z}+\varkappa\,\Delta M.$$

The equation of continuity now becomes

$$\varDelta L = -\frac{\partial M}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The three first equations lead to the two following:

$$c^2 \nu_3 = \text{Constant} + \mathfrak{z} - \varkappa \frac{\partial M}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\varDelta M = \frac{1}{\varkappa} \cdot \frac{\partial \mathfrak{z}}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the functions $L$ and $M$ are so determined that they satisfy the boundary conditions then the problem is to be considered as solved and equation (9) gives the desired distribution of pressure. As boundary conditions I have retained those previously laid down, viz, adhesion to the earth's surface, slipping on an upper boundary surface at an altitude $R.\ h$ above the earth whereby $h$ is to be considered as a small number in comparison with unity.

For further calculation it is expedient to introduce the vertical and meridional components of the current or $V$ and $N$. These are connected with $L$ and $M$ by the equations

$$\left.\begin{aligned} V &= \frac{\partial L}{\partial r} + M \cos\theta \\ N &= -\frac{1}{r}\frac{\partial L}{\partial \theta} + M \sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The equation of continuity now becomes

$$\frac{\partial V}{\partial r} + \frac{2}{r} V = \frac{1}{r}\left\{ \cot\theta \,.\, N + \frac{\partial N}{\partial \theta} \right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The elimination of $L$ gives the further equation

$$\frac{\partial (Nr)}{\partial r} + \frac{\partial V}{\partial \theta} = r \frac{\partial M}{\partial r} \sin\theta + \frac{\partial M}{\partial \theta} \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The calculation gives the following values:

$$V = \frac{2\varepsilon}{\varkappa} R^3 \left\{ \chi_1 + 2\chi_2 - 6\,(4\chi_1 + \chi_2)\cos^2\theta + 35\chi_1 \cos^4\theta \right\} . f(\sigma) \quad . \quad (14)$$

$$N = \frac{2\varepsilon}{\varkappa} R^3 \sin\theta\cos\theta \left\{ -\chi_1 - 2\chi_2 + 7\chi_1 \,.\, \cos^2\theta \right\} . \varphi(\sigma) \quad . \quad . \quad . \quad (15)$$

In these $f(\sigma)$ and $\varphi(\sigma)$ have a signification similar to that in the previous memoir, namely,

$$\left.\begin{aligned} f(\sigma) &= \frac{\sigma^2}{48}(h-\sigma)(3h-2\sigma) \\ \varphi(\sigma) &= \frac{\sigma}{48}\left\{ 6h^2 - 15h\sigma + 8\sigma^2 \right\} \end{aligned}\right\} (16)$$

Moreover, $\sigma$ is determined by the same equation as before,

$$r=R\,(1+\sigma)$$

Finally, from the equation (9)

$$c^2\nu_3=\text{const}+\mathfrak{F}-\kappa\frac{\partial M}{\partial z}$$

there results the following:

$$c^2\nu_3=\text{const}+\varepsilon R^2\left\{\left(\frac{3\chi_1}{7}+\chi_2\right)\cos^2\theta-\chi_1\cos^4\theta\right\}\quad . \quad . \quad (17)$$

This last equation allows of a direct comparison with the above-mentioned observations of the distribution of pressure.

IV.

The average values of the pressure of the air in the Southern Hemisphere are given in the following table (under the column of observations) as a function of the latitude.*

*Air pressure at the earth's surface.*

| Latitude. | Observed. | Computed. |
|---|---|---|
| ° | *mm.* | *mm.* |
| 0 | 758.0 | 758.0 |
| S. 10 | 759.1 | 758.9 |
| 20 | 761.7 | 760.5 |
| 30 | 763.5 | 762.0 |
| 40 | 760.5 | 760.5 |
| 50 | 753.2 | 755.3 |
| 60 | 743.4 | 747.1 |
| 70 | 738.0 | 738.0 |
| 80 | ...... | 730.9 |
| S. 90 | ...... | 727.2 |

These pressures are fairly represented by an expression of the form

$$p=p_a+a\cos^2\theta-b\cos^4\theta.$$

If we determine the constants $a$ and $b$ from the observed values for two different polar distances, for which I have used $\theta=50°$ and $\theta=20°$, then we obtain

$$p=758+31.295\cos^2\theta-61.094\cos^4\theta.$$

By the means of this formula the values given in the second column, under "computed," have been obtained.

* See A. Sprung, *Lehrbuch der Meteorologie*, p. 193; J. van Bebber, *Handbuch der Witterungskunde*, II, p. 136. [These figures are taken originally from Ferrel, "Meteorological Researches," I, 1880.—*C. A.*]

Furthermore, if we make the very probable assumption that the variations in pressure here considered depend exclusively on the movement of rotation, that therefore

$$p=p_a(1+\nu_3)$$

where $p_a$ represents the pressure at the equator, then is

$$\nu_3=\frac{p-p_a}{p_a}.$$

Therefore

$$\nu_3=\frac{\cos^2\theta}{758}\left\{31.295-61.094\cos^2\theta\right\}$$

$$=0.0413\cos^2\theta-0.0806\cos^4\theta \quad . \quad . \quad . \quad . \quad . \quad (19)$$

But the computation of $\nu_3$ had already given

$$\nu_3=\frac{\varepsilon R^2}{c^2}\cos^2\theta\left\{\frac{3\chi_1}{7}+\chi_2-\chi_1\cos^2\theta\right\}$$

wherein the appended constant can be omitted.

Hence, the two expressions for $\nu_3$ can be put equal to each other, and for the computation of the motion of rotation we obtain the two equations

$$\frac{\varepsilon R^2}{c^2}\chi_1=0.0806$$

$$\frac{\varepsilon R^2}{c^2}\left(\frac{3\chi_1}{7}+\chi_3\right)=0.0413$$

If in these we put

$$R=6379600^{\mathrm{m}};\ c=280^{\mathrm{m}};$$
$$\varepsilon=0.00007292$$

then we shall obtain

$$\chi_1=0.0292\ \varepsilon$$
$$\chi_2=0.0836\ \chi_1.$$

Hence, the relative angular velocity of the rotary motion of the air is

$$\chi=0.0292\ \varepsilon\left\{\cos^2\theta-0.0836\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This is small in comparison with $\varepsilon$, the angular velocity of the earth, therefore it nowhere leads to improbably large movements of the atmosphere. If we form the product $\chi_1 R$, we obtain for it the value 13.58 metres per second. But the true linear velocity corresponding to the rotatory motion is

$$O=\chi.\ R.\ \sin\theta.$$

The maximum value of this occurs at S. latitude 56° 27′ and amounts to 4.59 metres per second. From the S. pole to 16° 49′ S. latitude the average

value of the rotatory motion is positive, that is to say, directed toward the east; thence to the equator the value is negative, therefore directed toward the west.

These results can easily be combined with the conclusions of my previous memoir, according to which the motion of rotation can be considered as the sum of two terms that are of entirely different natures. Of the second term it was remarked especially that the current corresponding to it first attains sensible values at great altitudes. This therefore becomes at that altitude materially larger than the above deduced average value. The first term gave a movement entirely confined to the lower strata of the atmosphere: it is directed toward the east from the pole down to 35° latitude, but directed toward the west exclusively in the equatorial zone and less in velocity than the first component movement. The numerical computation leads to the same conclusion, since $\chi_2$ is small in comparison with $\chi_1$ Since from 35° of latitude down to the neighborhood of the equator there are two currents of opposite signs flowing over each other, therefore the place where the average movement of rotation is 0° will lie nearer to the equator than to 35°.

Therefore the conclusion of W. Siemens, which gave the first stimulus to the present investigation, has to be subjected to a modification only in so far as we must consider that the westward movement of the upper regions and higher latitudes has a predominance over the easterly movement of the lower regions and lower latitudes, because the former loses a much smaller fraction than the latter of its living force in consequence of friction.

The vertical and meridional components $V$ and $N$ are to be added to the corresponding components that were computed in my first memoir. The vertical component is positive at the equator and at the pole, it therefore gives an ascending current at both places, whereas $V$ is negative throughout a broad central zone. *Therefore at the equator the ascending current is strengthened, at the pole the descending current is enfeebled.*

The meridional component $N$ is zero at the surface of the earth at the equator; it is negative, *i. e.*, it is directed toward the south from thence to about 24° latitude; thence to the pole, where it is again zero, it has a northerly direction. Therefore in the tropics it strengthens the equatorial current and in higher latitudes it enfeebles it. Perhaps this explains the occurrence of northwest winds which frequently occur in the southern hemisphere between 50° and 60° south latitude.

Finally it may be remarked that the formula above used for the distribution of pressure agrees still better with the observations if a third term with a 6th power of cos $\theta$ is introduced. This term would also find its explanation by the analytical development, since the newly found meridional current should properly be again evaluated, in order to further compute the movements of rotation that are to be added

to the first approximation, and which will bring about a corresponding change in the formula for pressure.

In other words, by a series of approximations one seeks the true solution in a manner similar, for instance, to that used in the computation of mutual inductive effects of two conductors, in which computation we imagine the total influence developed into a series of individual influences of the first conductor upon the second and then again of the second upon the first, and so on. It is easy to foresee that the further prolongation of the computation must afford a corresponding term in the expression for the pressure. By this means the expression for the rotatory motion will suffer some change; still it is to be seen that the order of magnitude of this is already correctly established. After the execution of the further computations just indicated, I expect then to elaborate in a similar manner the average distribution of pressure in summer and in winter in order to determine more precisely the changes of the rotatory motion with the seasons. The formula above found is only to be applied with caution to the northern hemisphere, since in this hemisphere the fundamental condition that the temperature is a function of the geographical latitude applies much less truly than in the southern hemisphere.

XIV.

# A GRAPHIC METHOD OF DETERMINING THE ADIABATIC CHANGES IN THE CONDITION OF MOIST AIR.*

By Dr. H. HERTZ.

The theoretical meteorologist daily has to discuss considerations as to the changes of condition that take place in moist air that is compressed or expanded without the addition of any heat. Hence he desires to attain answers to these questions with the least possible expenditure of time, and he does not care to use any of the complicated formulæ of thermo-dynamics. Actually he generally uses the small practical table that Professor Hann communicated in the year 1874 (*Zeit. der Oest. Ges. f. Met.*, 1874, IX, p. 328). Still it appears that with at least an equal convenience one may attain a greater completeness if one makes use of the graphic method, and the table accompanying this paper presents an attempt in this direction. This contains nothing theoretically new except in so far as that it also completely considers the peculiar behavior of moist air at 0° C., which, so far as I know, has hitherto not been treated of.† In the following I will now in Section I, collect together the exact formulæ of the problem, since a complete collection of such appears to be wanting. Under Section II, the presentation of the formulæ by the graphic table is described. Finally under Section III, I explain completely, although purely mechanically, the application of the latter to a numerical example. If one follows this example with the diagram in the hand, one attains a judgment as to the use of the table and a knowledge of the method of using it without the necessity of going through the computations of Sections I and II.

## I.

In a kilogram of a mixture of air and aqueous vapor let $\lambda$ represent the proportional weight of dry air and $\mu$ the proportional weight of unsaturated aqueous vapor contained therein. Let the pressure of the mixture be $p$ and its absolute temperature be $T$. Our problem is: What conditions will the mixture pass through when its pressure is di-

* Translated from the *Meteorologische Zeitschrift*, 1884, vol. I, pp. 421–431.

† See, however, Guldberg and Mohn, "Studies on the movement of the atmosphere," part 1, pp. 9–16, and, also, by the same authors, *Oest. Zeit. f. Meteorologie*, 1878, XIII, p. 117.

minished indefinitely without addition of heat? We must distinguish different stages.

*First stage.*—The vapor is unsaturated; liquid water is not present. We assume that the unsaturated vapor follows the laws of Gay-Lussac and Mariotte. Let $e$ be the partial pressure of the aqueous vapor; $p-e$ be that of the dry air; $v$ the volume of a kilogram of the mixture. We then have $p-e=\lambda\frac{R\,T}{v}$; $e=\mu\frac{R_1\,T}{v}$ where $R$ and $R_1$ are constants of well known meaning and value.

Since now the total pressure $p$ is the sum of these two values, therefore

$$pv=(\lambda R+\mu R_1)\,T$$

and this is the so-called equation of condition [equation of elasticity] for the mixture. If further, $c_v$ is the specific heat of air at constant volume and $c^1_v$ the same for aqueous vapor, then in order to bring about the changes $dv$ and $dT$, the quantity of heat to be added to the air must be

$$dQ_1=\lambda\left\{c_v\,dT+A\,R\,T\frac{dv}{v}\right\}$$

On the other hand, the quantity of heat to be added to the aqueous vapor must be (see Clausius *Mechanische Wärmetheorie.* 1876, vol. I, p. 51.)

$$dQ_2=\mu\left\{c'_v\,d\,T+A\,R_1\,T\frac{dv}{v}\right\}.$$

Therefore for both together, the quantity of heat is

$$dQ=(\lambda\,c_v+\mu\,c'_v)\,d\,T+A\,(\lambda\,R+\mu\,R_1)\,T\frac{dv}{v}$$

But this quantity of heat must be zero for the adiabatic changes now investigated by us. In order to integrate the differential equation arising from putting $dQ$ equal to 0, we divide it by $T$. From the mechanical theory of heat we know beforehand that by this operation the equation becomes integrable, and we find this confirmed *a posteriori.* If we carry out the integration and eliminate $v$ by means of the equation of elasticity, in that we recall that $c_v+AR$ is equal to $c_p$ or the specific heat under constant pressure there follows

$$(\lambda\,c_p+\mu\,c'_p)\log\frac{T}{T_0}-A\,(\lambda\,R+\mu\,R_1)\log\frac{p}{p_0}=0\quad\ldots\quad(1)$$

The quantity that forms the left-hand side of this equation has a physical significance. It is the difference of the entropy of the mixture in the two conditions that are characterized by the quantities $pT$ and $p_0\,T_0$. Moreover the mixture evidently behaves exactly like a gas

whose density and specific heat have values midway between those of the aqueous vapor and the air.

We now have to compute the limit of $p$ up to which the equation (1) may be used. Hereafter let $e$ be the pressure of the saturated aqueous vapor at the temperature $T$; $e$ is a function of $T$, but of $T$ only. The mass $\nu$ of saturated aqueous vapor that is present in the volume $v$ at the temperature $T$ amounts to

$$\nu = \frac{ve}{R_1 T} \quad \ldots \ldots \ldots \ldots \quad (1a)$$

and this quantity must be greater than $\mu$ so long as the vapor is unsaturated. Therefore the limit occurs when $\mu = \nu$. If we substitute for $v$ its value from the equation of elasticity, then this latter condition ($\mu = \nu$) takes the form

$$p = \frac{\lambda R + \mu R_1}{\mu R_1} e \quad \ldots \ldots \ldots \ldots \quad (1b)$$

As soon as $T$ and $p$ attain values that satisfy this equation, we must relinquish the use of equation (1) and pass over to the equations for the second stage.

*Second stage.*—The air is saturated with aqueous vapor and contains also additional fluid water. We neglect the volume of the latter. We can therefore here also consider the air on the one hand and the water, with its vapor, on the other hand, each as though the other were not present. To both are to be ascribed the same volume $v$ and the same temperature $T$ as that of the mixture; on the other hand, the pressure $p$ of the mixture is equal to the sum of the partial pressures, $p_1 = \frac{\lambda RT}{v}$ for the air and $p_2 = e$ for the aqueous vapor.

The equation $p = \lambda \frac{R\,T}{v} + e$

or

$$(p - e)\, v = \lambda\, R\, T$$

is therefore now the equation of elasticity of the mixture. The quantity of heat that we must communicate to the air in order to bring about the changes $dT$ and $dv$ is as before

$$dQ_1 = \lambda \left\{ c_v d\,T + ART\frac{d\,v}{v} \right\}.$$

On the other hand, the quantity of heat that must be communicated to the water in order to bring about the change $dT$, and to simultaneously increase by $d\nu$ the quantity $\nu$ of aqueous vapor, while pressure and volume change correspondingly, is

$$dQ_2 = Td\left(\frac{\nu r}{T}\right) + \mu c dT.$$

This equation is deduced in Clausius *Mech. Wärmetheorie*, vol. I, section vi, art. 11; and in it $c$ is the specific heat of liquid water, $r$ the external latent heat of vapor, both of them expressed in units of heat. Therefore the total heat communicated to the mixture is

$$dQ = \lambda \left\{ c_v d\ T + ART\frac{dv}{v} \right\} + dT\left(\frac{\nu r}{T}\right) + \mu c dT.$$

Here also we have to put $dQ=0$, then divide by T and integrate. With the help of the equation of elasticity and equation ($1a$) we eliminate the quantities $v$ and $\nu$ from the integral equation, and thus obtain

$$\left(\lambda c_p + \mu c\right) \log\frac{T}{T_0} + \lambda AR \log\frac{p_0 - e_0}{p-e}$$
$$+ \lambda\frac{R}{R_1}\left\{\frac{r}{T}\frac{e}{p-e} - \frac{r_0}{T_0}\frac{e_0}{p_0-e_0}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Here also the quantity on the left hand that is equated to zero represents the difference of the entropies between the final and the initial conditions of a kilogram of the mixture. The equation thus obtained can be used until the temperature attains the freezing point, then we arrive at the third stage.

*Third stage.*—In this case, in addition to the vapor and the liquid water, the air contains also ice. By further expansion of the air, the temperature will now not sink immediately further, for the latent heat of the freezing water will, even without a lowering of temperature, furnish the force necessary for overcoming external pressure. But the heat of liquefaction must not be applied to this purpose only, but also to the evaporation into vapor of a part of the already condensed water. For since the volume increases during the expansion without allowing the temperature to sink, therefore at the end of the process again, more water is become vapor than before, therefore the weight of the ice that is formed will be less than that of the fluid that was present.

Let now, again, $\nu$ be that portion of $\mu$ that is in the form of aqueous vapor, $\sigma$ the part that exists as ice, and $q$ the latent heat of liquefaction of a kilogram of ice. $T$, $e$, $r$ are constants. Since therefore $dT=0$, we have now only to communicate to the air the quantity of heat $\lambda\ ART\frac{dv}{v}$ and to the water that we evaporate the quantity of heat $rd\nu$, and to the water that we allow to freeze the quantity $-qd\sigma$. Therefore the quantity of heat given to the whole mixture is

$$dQ = \lambda\ ART\frac{dv}{v} + rd\nu - qd\sigma.$$

If we put $dQ=0$, divide by $T$ and integrate, there follows

$$\lambda\ AR \log\frac{v}{v_0} + \frac{r}{T}(\nu - \nu_0) - \frac{q}{T}(\sigma - \sigma_0) = 0$$

The division by $T$ was necessary in this case only in order to give the left-hand side of the equation the form of a difference of entropy. With the help of the equation of elasticity and the equation (1$a$) we can eliminate $v$ and $\nu$, and introduce instead of them the pressure $p$. The equation then shows us how the quantity $\sigma$ of ice that is formed varies with the change of pressure. The details of this process however interest us less than the limits within which it takes place. Therefore we let the subscript index figure 0 refer to the condition in which the mixture just reaches the temperature 0° in which therefore ice is not present, and where $\sigma_0=0$. On the other hand we let the subscript index figure 1 refer to the condition in which the last particles of water are freezing, in which therefore the temperature just begins to fall below zero. In this condition, evidently, $\sigma=\mu-\nu$, since only ice and vapor are now present. If now we substitute these values after introducing the pressure, there results

$$\lambda\, AR \log \frac{p_0-e}{p_1-e}+\lambda \frac{R}{R_1}\cdot\frac{e}{p_1-e}\cdot\frac{r+q}{T}-\lambda\cdot\frac{R}{R_1}\cdot\frac{e}{p_0-e}\cdot\frac{r}{T}-\mu\frac{q}{T}=0 \quad . \quad . \quad (3).$$

This equation connects the pressures $p_0$ and $p_1$, at which respectively the third stage is attained and relinquished.

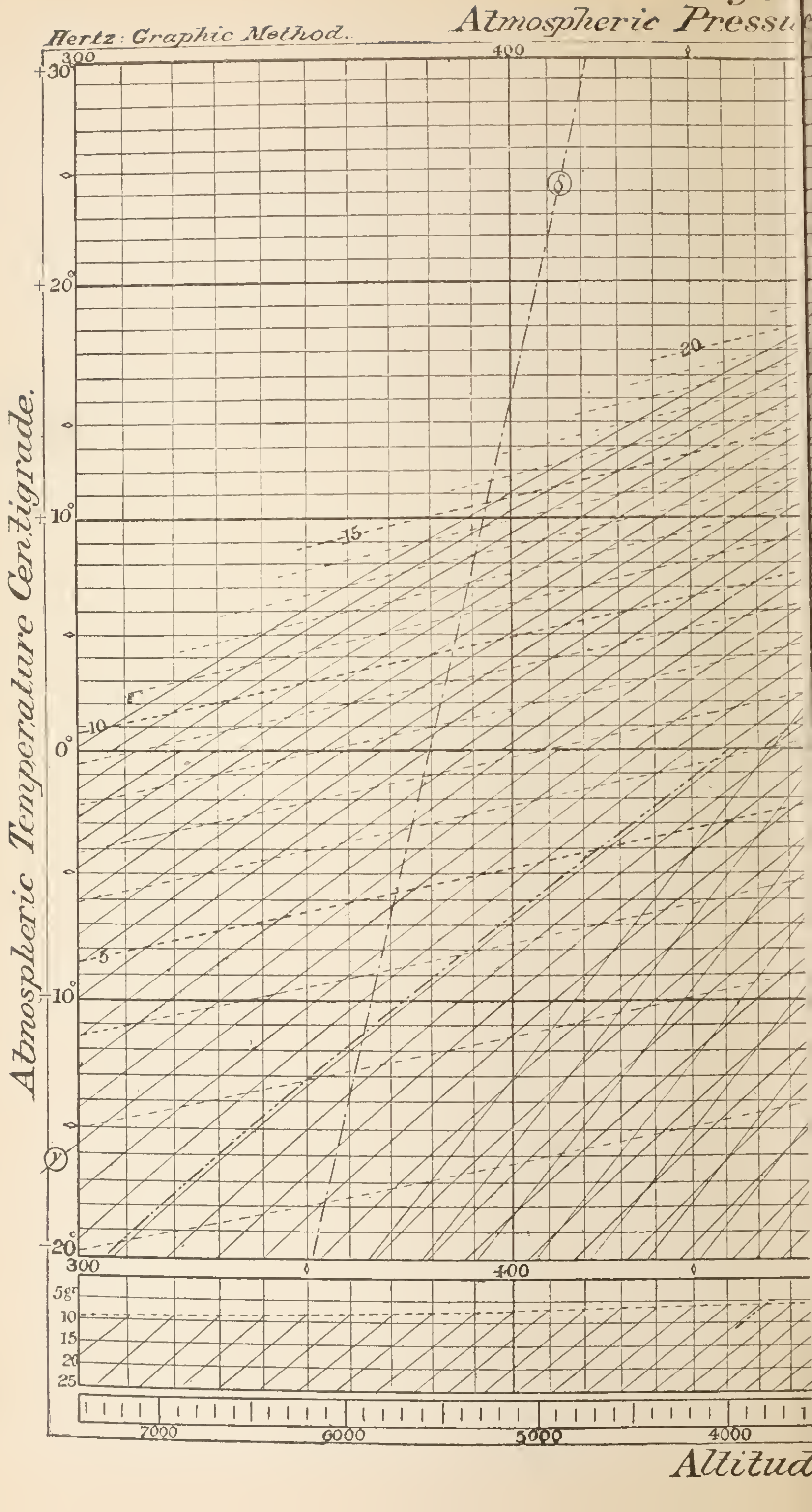
It was not necessary to append an index figure to the quantities $e$ and $T$ since they are alike for the initial and final conditions.

*Fourth stage.*—If now the temperature sinks lower, we have then only vapor and ice. The relations that we have to consider are the same as in the second stage, and the final formula is also the same. Only here the specific heat of evaporation has another value from that there given. Here, namely, it is equal to $r+q$ since the heat that is necessary to immediately change ice into vapor must exactly equal the heat that is needed to first melt the ice and then change the water into vapor. If we would be perfectly rigorous we ought not to assume $q$ as constant, but must consider it as slightly variable with the temperature, but the differences are so small that here they may remain out of consideration. In this fourth stage we may attain to those low temperatures at which the air itself can no longer be considered as a permanent gas.

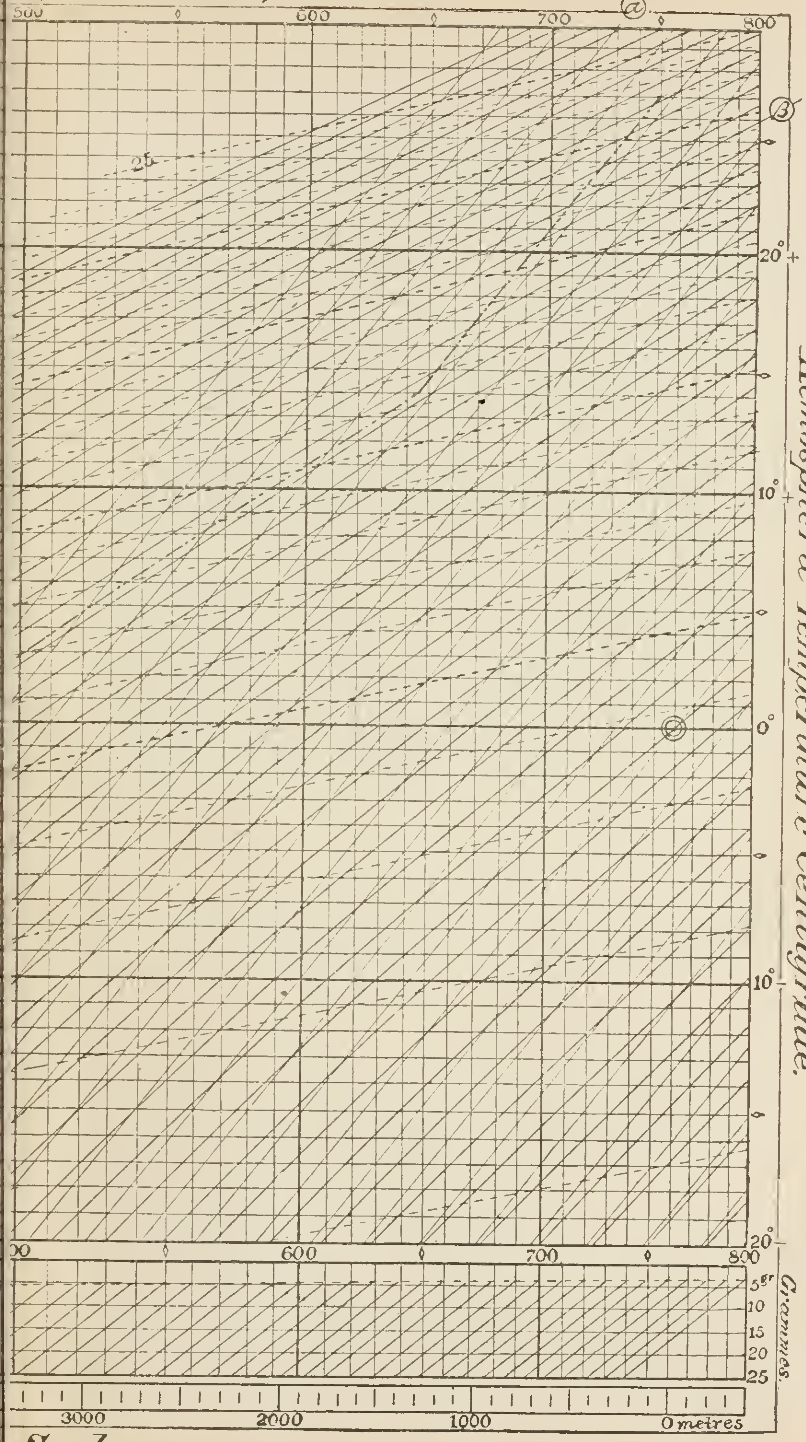
The four stages that we have here distinguished, one can very properly designate as the dry, the rain, the hail, and the snow stage.

If one is now in a position such that he is obliged to exactly follow the changes that a mixture containing a considerable percentage of water must undergo, then nothing further remains than to abide by these more complicated formulæ. In that case one proceeds in the following manner: First we substitute the values of $\lambda$ and $\mu$ in all the equations. Then we substitute the quantities $p_0$ and $T_0$ for the given initial condition in equation (1). We then consider the resulting equation and the equation (1$b$) as two simultaneous equations with the two unknown quantities $p$ and $T$. Solving those equations with reference to these quantities, we obtain that condition through which we must go in pass-

Fig. 2


Atmospheric Pressu
300
400
+30°
+20°
+10°
0°
-10°
-20°
Atmospheric Temperature Centigrade.
δ
γ
20
15
10
5
300
400
5gr
10
15
20
25
7000
6000
5000
4000
Altitud

e (Millimetres.)




Scale.

ing from the first to the second stage. The values thus obtained are then to be substituted as $p_0$ and $T_0$ in equation (2). By substituting $T=273°$ in the equation thus obtained, we obtain that $p_0$ which occurs in the equations of the third stage. If now we further determine from the equation (3) the final pressure $p_1$ of the third stage then this pressure and the temperature 273° form the $p_0$ and $T_0$ of the equations of the fourth stage. It will frequently happen that the temperature down to which the first stage holds good will lie below the freezing point; in that case one passes directly over to the fourth stage, omitting the second and third. After we have thus determined for all the equations the coefficients and the limits for which each equation holds good we can use them in order to determine the $T$ belonging to any given $p$ or inversely. All these computations can however only be executed by successive approximations, and one would do well to take the necessary approximate values from the accompanying diagram. If we have determined $p$ and $T$ for any special condition then the remaining characteristics are easily found. The density of the mixture follows from the corresponding equation of elasticity. The equation (1$a$) gives the quantity of water still present in the form of vapor, and therefore also the quantity of water already liquefied. Frequently one desires to know the difference in altitude $h$ that corresponds to the different conditions $p_0$ and $p_1$ under the assumption that the whole atmosphere is found in the so-called condition of adiabatic equilibrium. If one desires the exact solution of this problem, it must be attained by the laborious mechanical evaluation of the integral

$$h=\int_{p_1}^{p_0} vdp\,;$$

but since it is precisely with regard to this point that an exact determination never has a special value, therefore here one may always abide by the accompanying convenient diagram.

II.

If we had to deal only with one mixture whose composition is *exactly* known for which we therefore can have only *one* value of the ratio $\mu:\lambda$, then we could exactly re-produce the formulæ above developed by a graphic table that would enable us to directly perceive the adiabatic changes of the mixture for any condition.

We should represent pressure and temperature by coördinates in one plane and cover this plane with a system of curves that should connect all those conditions together that can adiabatically pass from one to the other. It would then only be necessary to glide from a given initial condition along the curve going through the corresponding point in order to perceive the behavior of the mixture as it passes through all these stages.

Since however the meteorologist must necessarily deal with mixtures

of very various proportions, therefore by this method a great number of tables would be required. But it can now be shown that one can also manage with only one graphic table, if first we confine ourselves to those cases in which the weight and pressure of the aqueous vapor is small in comparison with the weight and pressure of the air, and if secondly we do not require of the results any greater accuracy than corresponds to the neglect of those quantities in comparison with these. If we neglect $\mu$ and $e$ as compared respectively with $\lambda$ and $p$, then the form of the curves to be drawn is the same for all the absolute values of $\mu$, therefore the same curve can be used for all the different mixtures. But the points at which the different stages pass into each other will be located very differently for different mixtures, and special devices will therefore be needed by means of which this point may be determined. The graphic table, Fig. 28, is therefore constructed in accordance with the following principles.

The pressures are laid off as abscissas on the adopted scale for the interval between 300 millimetres to 800 millimeters of the barometer; the temperatures are laid off as ordinates for the interval between $-20°$ Cels. and $+30°$ Cels. But as we see by the diagram, a uniform increase in the length of either of these coördinates does not indicate an equal increase of pressure or of temperature; on the contrary the diagram is so constructed that an equal increase of distance corresponds to an equal increase in the logarithm of the pressure and in the logarithm of the absolute temperature. The advantage of this arrangement consists in the fact that thus the curves with which we have to do become, some of them exact, and some of them approximate straight lines, which brings an important advantage in the accurate construction and use of the table.

Now the adiabatics of the first stage (if we neglect $\mu$ with respect to $\lambda$) are given by the equation

$$c_p \log T - AR \log p = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

In this diagram the logarithms are always those of the natural system. With Clausius we put

$$c_p = 0.2375 \frac{\text{Calorie}}{\text{Cels. degree} \times \text{kilogr.}}$$

$$A = \frac{1}{423.55} \frac{\text{Calorie}}{\text{Kilogrammetre}}$$

$$R = 29.27 \frac{\text{Kilogrammetre}}{\text{Cels. degree} \times \text{kilogr.}}$$

These adiabatics appear in our diagram as straight lines. One of them is distinguished by the letter alpha ($\alpha$) and the whole of this system may be called by this letter. The individual lines are so drawn that

from one to the next the value of the constant (which is the entropy) increases by the quantity

$$0.0025 \frac{\text{Calorie}}{\text{Cels. degree} \times \text{kilogram}}.$$

These lines therefore appear at equal distances apart from each other. One of them is drawn to the point 0° Cels., and the pressure 760 millimetres.

The curves of the adiabatics in the second stage must satisfy the equation*—

$$c_p \log T - AR \log p + \frac{R}{R_1} \cdot \frac{r}{T} \frac{e}{p} = \text{constant} \quad . \quad . \quad . \quad . \quad (\beta)$$

In this equation $\frac{R}{R_1}$ is the density of aqueous vapor in reference to the air, and therefore is equal to 0.6219. According to Clausius,

$$r = 607 - 0.708 \,(T - 273) \frac{\text{calorie}}{\text{kilogram}}.$$

I have taken the value of $e$ for the different temperatures from the table computed by Broch (*Travaux. du Bur. Internat. des Poids et Mesures*, tome I). The curves run along with feeble curvature from the right hand above to the left hand below. One of these is distinguished by the letter beta ($\beta$). They also are so drawn that the entropy increases from one to the next by a constant value of—

$$0.0025 \frac{\text{Calorie}}{\text{Cels. degree} \times \text{kilogram}},$$

or the same as before for the alpha system, and so that one of them passes through the point 0° C., 760 millimetres.

The curves that correspond to the third stage coincide with the isotherm of 0° C.

Finally the curves of the fourth stage are entirely similar to those of the second stage, but are not exactly the same, for their formula is derived from that belonging to the second system by substituting $r + q$ for $r$ where $q$ is equal to 80 calories per kilogram. They are distinguished by the letter gamma ($\gamma$), and are drawn according to the same rules as alpha ($\alpha$) and beta ($\beta$) curves. In general the gamma curves are not precise prolongations of the $\beta$ system.

We have now to find some means by which the points of transition

---

* Although $\mu$ is neglected in comparison with $\lambda$, still it is questionable whether $c\mu$ is negligible in comparison with $c_p\lambda$, since $c$ is four times larger than $c_p$. Even although within the limits of the diagram $\mu$ does not exceed $\frac{1}{40}\lambda$, yet the $c\mu$ is $\frac{1}{10} c_p\lambda$. But in meteorologic applications we recall that in these extreme cases the liquid water is not generally wholly carried up with the air. Frequently so large a fraction of it falls from this air as rain that we keep nearer the truth when we entirely neglect the specific heat of the liquid water, rather than to introduce it with full value into the computation.

can be found for the different stages. The dotted lines serve to show the end of the first stage. These lines give the greatest quantity of water, expressed in grams and computed according to the formula $\nu = \frac{R}{R_1} \cdot \frac{e}{T}$ that a kilogram of the mixture in the different conditions can contain as vapor. Thus, for instance, the curve designated by 25 connects all those conditions in which one kilogram of the mixture when saturated contains 25 grams of vapor. These curves are drawn from gram to gram. If a mixture contains $n$ grams of vapor in every kilogram of mixture, then evidently we have to follow the curve of the first stage up to the dotted line $n$, but then we must pass either to the second or fourth stage.

The limit of the second stage, with respect to the third, is given by the intersection of the corresponding adiabatic beta with the isotherm of 0° C. By the pressure $p_0$, that corresponds to this intersection, and by the quantity $\mu$ of water, is determined the pressure $p_1$, at which the transition takes place from the third to the fourth stage. The small auxiliary diagram that is given beneath the main table of Fig. 28 serves for the graphic determination of $p_1$. This auxiliary diagram contains as abscissa the pressure arranged as in the larger diagram, and as ordinate the total quantity $\mu$ of the water in all conditions expressed in grams per kilogram of the mixture. The oblique lines of this small table are the curves that correspond to the equation (3) of the third stage, when in this equation we consider $p_0$ as constant, but $p_1$ and $\mu$ as the variable coördinates. These lines are not perfectly straight, but are not to be distinguished from such in a diagram on this scale. The highest point of each of these lines corresponds to the case $p_1 = p_0$. The corresponding $\mu$ is not zero, but is equal to the least value, $\nu$, that $\mu$ must have in order that the mixture may be saturated at 0° C., and the auxiliary table come into use. If one wishes to find the $p_1$ belonging to a definite value of $p_0$ and $\mu$, then we seek that oblique line whose highest point lies on the abscissa $p_0$, and then we pass along this line downwards to the ordinate $\mu$. The pressure at which we attain this ordinate is the desired pressure $p_1$. In this pressure we have the point of transition from the third to the fourth stage.

Having in this way determined the totality of the stages through which the mixture runs, we find the remaining desired quantities for each stage in the following manner:

(1.) The dotted line which one selects, (corresponding to the condition given,) indicates directly the number of grams of water still remaining in the form of vapor. If we subtract this quantity from the original total quantity $\mu$, we obtain the quantity of water that has already been condensed.

(2.) The density $\delta$ of the mixture can under the adopted approximations be computed for all conditions by the formula

$$\delta = \frac{p}{RT}; \text{ or } \log \delta = \log p - \log T - \log R.$$

These can also be read off graphically if the diagram is covered with another system of lines of equal density. We see that these lines will constitute a system of parallel degrees of density.

Only one of these lines is in reality drawn on the accompanying diagram, namely, the line marked $\delta$ (delta), in order not to confuse the diagram. But with the assistance of this one we can also compare the densities in any two conditions $C_1$ and $C_2$, according to the following rule: From the points 1 and 2, representing these conditions on the diagram, draw two straight lines, respectively, parallel to $\delta$, until they intersect the isotherm 0° C., and read off the pressures $p_1$ and $p_2$ for these points of intersection. The densities for the conditions $C_1$ and $C_2$ are in the ratio of the pressures $p_1 : p_2$; as is seen from the considerations that the densities for the condition ($p_1$, 0°), and for ($p_2$, 0°) are according to Mariotte's law in the ratio of $p_1$ to $p_2$, and are equal to the densities for the conditions $C_1$ and $C_2$ since they lie on the same line of equal density with these.

(3.) The difference of altitude $h$ that corresponds under the assumption of adiabatic equilibrium to the passage from the condition $p_0$ to the condition $p$ is given by the equation

$$h = \int_p^{p_0} v dp = R \int_p^{p_0} T \frac{dp}{p}.$$

In using this equation we take $T$ as a function of $p$ from the diagram and then perform the integration mechanically. Actually however the assumption of adiabatic equilibrum is always so imperfectly fulfilled that it is not worth while to trouble about an exact development of its consequences. On the other hand, for moderate altitudes, we commit a relatively very unimportant error when we give $T$ an average value, and consequently consider it as constant. Within the limits of the diagram $T$ ranges only between the values 253 and 303; if therefore we give it the constant value $T_0 = 273$, then the error in $h$ will scarcely exceed one-ninth of the whole value. If we are satisfied with this error, then we have

$$h = \text{constant} - RT_0 \log p,$$

and we now can, along with the pressure, directly introduce the altitude as abscissa. Consequently an equal increase in the length of the abscissa will everywhere correspond to an equal increase in altitude. The scale of altitudes is introduced at the base of the diagram. Its zero point is put at the pressure 760, because this is usually taken as the normal pressure at sea-level.

III.

In order to illustrate the use of the table by an example, we propose to ourselves the following concrete problem: Given a mass of air at sea-level under the pressure of 750 milimetres, the temperature 27 degrees

centimetre, and relative humidity 50 per cent., it is desired to find what conditions this mass of air will pass through when it is carried without change of heat into the higher strata of the atmosphere, and therefore into a lower pressure, and at what approximate altitudes above the sea-level the different conditions will be attained.

We first seek from the diagram the point that corresponds to the initial stage. We find it as the intersecting point of the horizontal isotherm 27 and the vertical isobar 750. We remark that it lies almost exactly on the dotted line 22. This indicates that our mass of air must contain 22.0 grams of aqueous vapor in each kilogram of its own weight in order to be saturated. Since however it has only a relative humidity of 50 per cent., therefore it contains 11.0 grams of water per kilogram. We note this for future use. Furthermore, we go along down the isobar 750 to the scale of altitude that is found at the lowest edge of the diagram, and here we read off 100 metres. The 0 point of the scale of altitude therefore lies about 100 metres below the sea-level adopted by us as a base, and therefore we have to subtract 100 metres always from all the direct readings on the altitude scale, in order to obtain the altitude above sea-level. If now we raise our atmospheric mass upward, then the series of conditions which it runs through will be directly given by that line of the Alpha system that passes through the initial condition.* An engraved line not being given for this case we therefore interpolate such an one (*i. e.*, the — . . — . . line of the diagram). If the number of intersecting lines appears to be bewildering, then we take a strip of paper and lay it parallel to the system under consideration, when all confusion disappears. In order now to recognize the condition in the neighborhood of the altitude 700 metres we seek for the point $700 + 100 = 800$ on the scale of altitudes, and go perpendicularly up until we intersect our Alpha line. The intersection gives this point at pressure 687 milimetres, and temperature 19.3° C. But we ought to use the Alpha line only to that point in which it itself intersects the dotted line 11 (or the line of absolute weight of contained water). The attainment of this line indicates that we have arrived at a condition in which the air is only just able to contain 11 grams of water per kilogram in the form of aqueous vapor. Since now we have 11 grams per kilogram, therefore with any further cooling condensation begins. The pressure for the point at which precipitation commences is 640 milimetres; the temperature is 13.3° C. This is not the temperature of the original dew-point, but it is lower. The dotted line, eleven, intersects the isobar 750 at 15.8° C., and this is the initial dew-point. But since besides cooling our air has also experienced an increase in its volume, therefore the vapor has remained volatile to a

*The letters $\alpha$, $\beta$, $\gamma$, that designate the systems are to be found in the small circles at the edge of the diagram. For each of these there corresponds one line of the system that it designates. A line of special dots and dashes in the diagram indicates the change of condition of the air in our illustrative example.

temperature 13.3. The altitude at which we now find ourselves corresponds to the lower limit of the formation of clouds, and is about 1,270 metres. In order to follow the conditions further we draw a curve of the Beta ($\beta$) system through the point of intersection.

This curve is inclined much more slowly toward the axis of abscissas than the Alpha line hitherto used, therefore the temperature now changes with the altitude much more slowly than before, which is due to the evolution of the latent heat of the aqueous vapor. We have now risen 1,000 metres since the commencement of condensation, but the temperature has sunk only to 8.2°, or only 0.51° to each 100 meters. We now find ourselves on the dotted line 8.9, and perceive that 8.9 grams of water are still in the state of vapor; that therefore in this first 1,000 metres of the cloud layer 2.1 grams of water have been condensed per kilogram of air. We attain the temperature zero degrees C. at the pressure 472 millimetres, and at the altitude 3,750 meters, whereas if the air has been dry, and we had not been obliged to leave the Alpha line, this temperature would have been attained at an altitude of 2,600 metres. It now appears that by this time 4.9 grams of water, or 0.45 per cent. of the total contents, have been condensed, and during further expansion this portion begins to freeze and form hail [the reader will recall that although 45 per cent. has been condensed into visible cloud, yet it has not separated from its original air and been precipitated as rain, but is still rising with the air and of course cooling with it]. But the temperature can not sink further until the last particle of water is frozen, and we therefore must retain the temperature 0° uniformly during a certain distance of further ascent.

In order to ascertain this distance we make use of the auxiliary diagram between the scale of altitude and the larger diagram, we pass down the isobar 472 millimetres to the dotted line of this diagram; we draw through this intersection a line parallel to the inclined line of the auxiliary table, and go along this line until we reach that horizontal line that is characterized by the number 11, or the total weight of the contained water, and which we easily interpolate between the engraved lines 10 and 15. As soon as we have attained this line we read off the pressure $p = 463$ millimetres, and turn back to the larger diagram. At the pressure thus found the process of freezing is finished, and the layer within which it all takes place has a thickness of about 150 metres. It must surprise one that, according to the dotted line, the quantity of water in the form of aqueous vapor has again increased a little during the process of freezing. But this is quite correct; in fact, the volume has increased without lowering the temperature. We leave the temperature 0° C. at the pressure 463 millimetres. The water which hereafter is precipitated passes directly over into the solid condition. Since there is now but little water as aqueous vapor, therefore the temperature again begins to sink more rapidly with the altitude. We ascertain the different conditions in that we make use of

that special Gamma line that can be drawn through the point 463 millimetres on the isotherm 0° C. The temperature −20 down to which our table can be used is attained at the altitude 7,200 metres, and at the pressure 305 millimetres, at which only two grams of water per kilogram remain as vapor, the other nine having been condensed. If it interests us to know how the density in this condition is related to the density in the initial condition, we draw through the corresponding points two lines parallel to the Delta line. These intersect the isotherm of 0° C. at the pressures 330 and 680 millimetres. The densities are to each other as these pressures, namely, as 33 to 68; and as 33 and 68 are to 76, so they are related to the density of the air in its normal condition of 0° C. temperature and 760 millimetre pressure.

All these items are directly read off from the diagram. Errors that could be injurious certainly occur only in the altitudes. These latter refer strictly speaking to ascent in an atmosphere of a uniform temperature of 0° C. But it would have been generally better to have assumed that the temperature of the atmosphere is everywhere the same as that of the ascending mass of air. The resulting error can be materially reduced by a very little computation. Thus we found that the point where condensation began, is at the pressure 640 millimetres. To this corresponds an altitude of 1,270 millimetres, provided that the temperature is 0°, but in our case this is between 27° and 13°, therefore on the average about 20°. For this temperature the altitude must be about $\frac{20}{273}$ or $\frac{1}{14}$ greater, since the density of the air is by this same fraction smaller than for 0°. Therefore the altitude really lies between 1,350 and 1,400 millimetres.

We must still supplement the above example by the mention of special cases:

(1) We assumed in the above that during the hail-stadium the total quantity of water originally present in the air, namely, 11 grams, was still contained therein. This will certainly only be an appropriate assumption in the case of very rapid ascents. In other cases perhaps the greater part of the condensed water falls as rain, and therefore only a fraction of it remains to be frozen. If one has any estimate as to how great this fractional part is, then the diagram will always allow us to ascertain the correct conditions. Thus if in our example one had reason to assume that half of the water condensed at 0° were removed, then on attaining the isotherm of 0° only 8.5 grams of water per kilogram of air would be present. We should then in using the auxiliary table not descend to the horizontal 11, but only to the horizontal 8.5, and should have started from the temperature line of 0° at the point corresponding to the pressure 466 millimetres (instead of 463 millimetres); this would have been the only difference.

(2) If we had assumed not 50 per cent. but 10 per cent. relative humidity in our example we should then have been able to use the Alpha line only to the dotted line 2.2. This point of intersection occurs

at pressure 455 millimetres, and at temperature − 13.6° C., therefore considerably below 0. Therefore there would have been no formation of liquid water and therefore no stadium for the formation of hail but only sublimation of water from the vaporous into the solid condition. We should then from the intersection of the Alpha line with the dotted line 2.2 have followed directly the line of the Gamma system that might have passed through this intersecting point.

The question is not uninteresting—what dew point is the highest that our mixture could have possessed in its initial condition as to pressure and temperature, in order that the condensation of liquid water, that is to say, the condensation at temperature above 0° C. should be just avoided? In order to answer this we follow the Alpha line to the isotherm 0° and here find the dotted line 5.25. We therefore at the highest could have had 5.25 grams of water per kilogram of air. In order now to ascertain at what temperature the air would then have been saturated under a pressure 750 millimetres, we slide along the line 5.25 up to the isobar 750 and intersect it at the temperature 4.8° C., and this is the desired maximum value of the dew point.

KIEL, *October*, 1884.

XV.

# ON THE THERMO-DYNAMICS OF THE ATMOSPHERE.*

## (*FIRST COMMUNICATION.*)

By Prof. WILHELM VON BEZOLD.

In the application of the mechanical theory of heat to the processes going on in the atmosphere we have hitherto almost exclusively confined ourselves to those cases in which one can disregard the increase or loss of heat during the expansion or compression.

The so-called convective equilibrium of the atmosphere, the unstable equilibrium in cyclones, the phenomena of the *foehn* winds have all hitherto been treated of under the assumption that we have to do with adiabatic changes of condition.

In fact, especially in the last-mentioned phenomena, the quantity of heat used or produced by expansion and compression as also by the changes in the physical condition of the water, are so prominent in comparison with those that, in these rapidly executed processes, are introduced or taken away by other sources that the above-mentioned assumption may be said to be thoroughly allowable. In the investigation of the convective equilibrium we obtain, under this assumption, at least a glimpse of the special case that lies as a limiting case between the two greater groups that correspond to the loss or increase of heat. Notwithstanding these extremely restrictive assumptions, still through the above-mentioned investigations, the comprehension of meteorological processes has been furthered to such an extent that we must consider their introduction as one of the characteristic features of modern meteorology. But the more valuable are the results that are already attained in this manner, so much the stronger must be the desire to free ourselves from the above-given limitations, and to extend the application of the mechanical theory of heat to those atmospheric processes in which the increase and diminution of heat from without can be no longer neglected. That this generalization had not already been long before taken is certainly because the formulæ are extremely complicated, so

---

* Translated from the *Sitzungsberichte der König. Preuss. Akademie der Wissenschaften zu Berlin:* Berlin, April 26, 1888, pp. 485–522.

that one always runs in danger of losing the leading thought in the midst of the notation and signs.

But in consideration of the fundamental importance that the application of the mechanical theory of heat in the most comprehensive manner possesses for the development of meteorology, one evidently ought not to be frightened by these extreme difficulties. This has induced me to make the attempt to introduce a method into meteorology that has proved so remarkably fruitful in the application of the mechanical theory of the heat to the theory of machines: I mean the graphic method that Clapeyron* has invented in order to make the ideas first expressed by Sadi Carnot† visible and comprehensible. Already, some years ago, a step in a similar direction was taken by Hertz‡ in a highly meritorious work on a graphic method for the determination of the adiabatic changes in moist air; but the problem that Hertz had before him, as also the method which he adopted, were materially different from those that I have now in mind. On the one hand, Hertz confined himself, as his title states, exclusively to the consideration of the adiabatic changes, and on the other hand, his object was only by means of a simple graphic process to avoid the complicated computations that one has to execute in following these changes. My object, on the other hand, has been to give a method of presentation that can serve as a guiding thread in the still more complicated formulæ with which one has to compute as soon as we disregard the restrictive assumption of adiabatic change, and that also allows one to draw certain important conclusions even from the form of the geometrical figures. To attain these objects however, scarcely any mental presentation is so appropriate as that introduced into science by Clapeyron, of course with such extensions as are required by the condition that in meteorological problems we have not as there to consider only two independent variables, but three, or in special cases, even still more.

But before I enter upon the subject itself I must touch upon another point on which notwithstanding its fundamental importance, remarkable to say, still perfectly clear views do not prevail. This has respect to the true reason of the cooling that occurs in the ascent of air to higher regions as well as the corresponding warming for descending air. While Sir William Thomson,§ Reye,|| Hann,¶ Peslin,** and with these investigators probably also the greater part of all physicists and meteorologists, correctly consider the cooling of ascending air as a consequence of the expansion occurring therein, on the other hand, Guldberg and

* Poggendorff's *Annalen*, vol. 59, pp. 446–566.

† *Réflexions sur la puissance motrice du feu.* Paris, 1824.

‡ *Meteorologische. Zeit.*, 1884, I, pp. 421–431. [See No. XIV of this collection.]

§ *Proc. of Manchester Soc.*, 1862, II, 170–176.

|| *Die Wirbelstürme*, Hannover, 1872.

¶ *Zeitschrift d. Oesterr. Ges. f. Met.*, 1874, Bd. IX, pp. 321, 337. Smithson. Rep. 1877, p. 397.

** *Bull. hebd. de l'Assoc. scientif. de France*, 1868, Tome III, p. 299.

Mohn* find the reason therefor in the work that is done in raising the air, and that is balanced by an equivalent quantity of heat taken from the air. Since by both methods of consideration the same value is found for the diminution of temperature with the height, therefore in the well-known excellent treatise of Sprung † both methods of consideration are presented beside each other as equally proper. But in fact only the first of these two is allowable, while that of Guldberg and Mohn contains in itself an error as to which one can only wonder that it could have escaped two such thoughtful investigators, and evidently also has hitherto not been remarked by others.

In order to obtain perfect clearness on this point one must first recall how it is that the ascending and descending currents in the atmosphere come to exist at all. This is, however, always brought about by differences in specific gravity that cause an ascent at certain places, while a corresponding mass descends at other places. The work that is required to raise the air at the one place is therefore always obtained by the falling of an equally great mass at another place. If no friction occurs the corresponding rising and falling movements once started would continue without any further addition of energy to infinity, and such an external addition of energy is only needed in order to overcome these frictions. These latter, however, are left out of consideration in all the discussions that are here considered, and this will also be done in the present memoir. We can consequently then compare the process with which we have to do, with movements in closed systems of tubes, such as a closed series of hot water pipes, or the movements of a continuous chain that hangs freely upon a roller. But it would never occur to any one to consider that the ascending water in the warmer half of a conduit, or the ascending portion of an endless chain must cool because of the work done in raising it. Similarly in the case of the ascending or descending currents in lakes or in the ocean, we must expect cooling or warming in consequence of these motions, if the ascent is accomplished at the expense of the heat latent in the fluid. The temperature changes occurring in the vertical motions of the air are therefore exclusively to be attributed to the work of expansion and compression, which is to be done or acquired respectively, and they would occur to precisely the same extent if the corresponding changes in pressure and volume occurred within a horizontal cylinder where rising and sinking was entirely out of the question.

On the other hand if we have air compressed within a vertical cylinder whose base is fixed, but which is closed above by a movable piston, and if we should now by a proper change in the load cause an expansion of the air then, besides the work of expansion, it would be necessary also to consider the work necessary in order to raise the center of gravity of the inclosed mass of air, and thus the cooling would be more

* *Zeit. Oesterr. Ges. Met.*, 1878, XIII, p. 113.

† *Lehrbuch d. Meteorologie*, Hamburg, 1865, p. 162.

considerable than when the whole change of condition took place with a horizontal position of the cylinder.

If the piston were without weight and without any loading, and if it were only at the beginning held fast but then suddenly loosed, and first held fast again at some other position at a greater distance from the base, then indeed the cooling would be attributable alone to the work which was necessary to be done in order to raise the center of gravity of the mass of air, since in this case no work of expansion is accomplished. By the explanations that I have made in such detail, in consideration of the fundamental importance of the question, it certainly ought to be perfectly clear that the cooling and warming in ascending and descending currents of air in the atmosphere are to be considered only as consequences of the work of expansion and compression; not of the work that is consumed in raising the air or that is gained by its descent, unless the ascending and descending masses belong permanently to one system. Since however the work of expansion and compression ought never to be left unconsidered, therefore in Guldberg and Mohn's method of consideration these, under all conditions, should have been further taken into consideration, and there would then have resulted for the rate of change of temperature with altitude a value exactly double that given by them. This being premised I will now pass to the problems mentioned in the opening paragraphs.

For our purpose it is first necessary to establish the fundamental quantities that come into consideration in investigations into the change of condition of a mixture of air and water or aqueous vapor. If in this I do not accord wholly with the steps that Hertz has chosen, this is because he has made various simplifying assumptions that are appropriate to the attainment of the end that he had in view, but that are not allowable in the general theoretical investigation that I contemplate. For the same reason I must again review the equations for the various conditions through which the mixture of air and water can pass, and which Hertz has developed in such a perspicuous manner, since not only by reason of the somewhat different notation, but also by the consideration of certain points intentionally neglected by Hertz, some material differences result.

Hertz and others in their investigations have made the assumption ordinarily used in the mechanical theory of heat that the unit of mass of the substance under consideration is given, and that it in succession passes through the different conditions. This assumption can not be rigorously adhered to in the case of atmospheric processes. A kilogram of moist air retains unchanged its mass only so long as during the expansion no condensation of aqueous vapor occurs, but suffers a diminution as soon as the formation of precipitation begins and rain, snow, or hail falls from it. When therefore a mass of moist air that is rising within a depression, or on the windward side of a mountain during a foehn on the lee side, is followed on its way through the atmos-

phere until it finally, either within an anti-cyclone or under the well known conditions of the foehn wind on the lee side of a mountain, comes again to its initial level, it is not the whole mass that we again find there present, but only a portion, although it may be a very considerable fraction, since a part of the water has been lost.

One can therefore begin the computation with the unit of mass of the mixture, but must consider the loss in mass that may occur in the course of the processes (a gain only occurs when the air passes over moist surfaces). But in this we have to combat the difficulty that, according to the point of departure that we choose, or according to the prevailing absolute humidity of the air at the point of departure, we have present, not only different quantities of vapor, but also different quantities of dry air, since the sum of the two must be equal to unity. It is therefore more appropriate to consider the unit of mass of dry air as given, and the water as an additional variable mixture.

This being premised, we will now indicate by $M_a$, $M_b$, $M_c$, $M_d$ the masses of the mixture in the four stadia so well distinguished by Hertz, namely, *the dry*, *the rain*, *the hail*, and *the snow* stage, and will also attach to the other quantities similar subscript letters as indices, in so far as a distinction of the respective stages may be necessary. But in computations that relate throughout to only one stadia these indices may be dropped, in order not to overburden the formulæ too much. This being premised, we next find for the four stages the accompanying equations that may be temporarily designated as the equations of mixture.

(*A*).—The dry stage:

$$M_a = 1 + x_a$$

or abbreviated

$$M = 1 + x$$

where $x$ or $x_a$ designates the mass of aqueous vapor that is mixed with the unit mass (one kilogram) of dry air. In this it is assumed that the air is not saturated with aqueous vapor, and therefore $x_a$ indicates always the mass of unsaturated (overheated) vapor that is contained in the mixture. This mixture remains, in general, constant in the free atmosphere, since in this stage precipitation is excluded and an appreciable introduction of aqueous vapor is only possible at the surface of the earth, and again since the quantity of aqueous vapor that is exchanged in the atmosphere between masses of air of different absolute humidities can certainly at first be wholly neglected.

(*B*).—The rain stage.

$$M_b = 1 + x_b + x_b'$$

or, when confined to one stage as before,

$$M = 1 + x + x'.$$

In this $x_b$ indicates the mass of saturated aqueous vapor that is contained in the air, $x'_b$ is the additional mass of water liquid that is present.

If we assume that by cooling, as for example through adiabatic expansion, the air has passed from the dry stage to the rainy stage, then will

$$M_b \overline{\overline{<}} M_a$$

wherein the equality sign is the limiting case but in general the inequality is to be considered as the characteristic sign. The quantity $x'_b$ is always very small and can only assume a somewhat greater value in exceptional cases, as for instance in the case of a remarkably strong ascending current of air that hinders the fall of the rain or rather that carries the drops upward with itself. How large this value may become we have as yet no indications whatever.

(*C*).—The hail stage: for this case

$$M_c = 1 + x_c + x'_c + x''_c$$

wherein $x_c$ is the quantity of saturated vapor; $x'_c$ the quantity of water present in the fluid condition; $x''_c$ the quantity of ice that is present. Here as above, under the corresponding assumptions, we have

$$M_c \overline{\overline{<}} M_b$$

this stage can, in general, only occur when fluid water is mixed with the air and this mixture is cooled to 0° Cent.

(*D*).—The snow stage; for this case

$$M_d = 1 + x_d + x''_d$$

where the notation is easily understood by what precedes and where again so far as the mixture can be considered as coming from the previous stage, we must have

$$M_d \overline{\overline{<}} M_c.$$

In the most common case, where an ascending mass of air $p$ by cooling gradually goes through all the different conditions, $x'$ and $x''$ are generally exceedingly small, so that the hail stage is entirely passed over, and in all formulæ only one independent variable $x$ appears. In this case $M$ steadily diminishes.

Hertz in his investigation has not considered the change of $M$, but has considered this quantity as constant. This was allowable in view of his object, but here as already stated in the beginning, this limitation must be avoided. The present more general consideration leads first of all to the recognition of the fact that here we have to do with a class of processes which so far as I know have not yet been considered in the mechanical theory of heat; such namely, as are reversible in their smallest parts but are not reversible as a whole.

So long as the quantities $x'$ and $x''$ are not equal to zero but possess a finite value even though exceedingly small, then can the quantity of

vapor that is condensed by cooling, as in expansion, be again evaporated by warming or compression. But as soon as the small quantity of water is evaporated, then by a further warming, the air enters again into the dry stadium but with a different quantity of vapor than it originally had, so that now it will pass through other conditions than at first, when the air expanded under continued loss of water. In order now to be able to determine perfectly the condition of the mass of air, we need beside the variables that occur in the equations of mixture to know also the volume $v$ that the mass $M$ occupies and the pressure $p$.

The latter we measure by the pressure in kilograms per square metre, wherein we now have to understand by kilogram the weight that a kilogram of mass has at 45° Lat. The simple relation

$$p = 13.6\beta$$

exists between the pressure $p$ thus measured or the so-called specific pressure and the barometric pressure $\beta$ expressed in millimetres of mercury; whereas expressed in atmospheres it has the value

$$P = \frac{p}{10333}$$

so that one can without difficulty pass from one mode of measurement to the other.

This much being prefaced, we can now establish the equations for the gaseous condition [equations of elasticity] for the different stages. Their general form is

$$f(v, p, t, x) = 0$$

therefore they contain one variable more than we generally find in the equations of elasticity. The quantities $x'$ and $x''$ do not appear in these since in general they are so small that they exert no influence on $p$ and $v$.

If now we would geometrically picture a condition of mixture we must (besides $p$ and $v$ which will be represented in the ordinary method by ordinates and abscissas in a rectangular system of coördinates with the axes $OP$ and $OV$) make use further of a third coördinate; as such we advantageously choose the value of $x$, and lay this off parallel to the axis $OX$ in a direction perpendicular to the plane $PV$. In this method of presentation, all conditions that correspond to any value of $x$ find their representation in one and the same plane, which only slightly differs from the $PV$ plane if we adopt the atmospheric pressure as the unit of pressure, and adopt lines of equal length in the direction of the axes of $V$ and $X$ as expressing the units of volume (one cubic metre) and of mass (one kilogram).

If now we imagine successive planes lying above each other, on which conditions are represented that differ progressively from gram to gram (that is, by a thousandth of the adopted unit), then these will lie

like sheets above each other, and in the study of the changes in condition one can simply adhere to the consideration of the curves described by the projection of the represented points upon the $PV$ plane. Therefore, this plane will frequently hereafter be briefly designated as the coördinate plane. We can therefore execute the mental presentation of these processes in this plane, if only certain artifices are used, of which mention will be made hereafter, and when we consider the resulting curves after a manner similar, as it were, to the lines on a Riemann surface. The most important result is, that thereby the external work consumed or expended finds its mental representation precisely as in the simple method of Clapeyron. The formula

$$d\,Q = A\,d\,U + A\,pdv$$

expresses the quantity of heat to be added for an infinitely small change of condition, under the notation* here adopted, and the special assumptions here considered; or if we pass from an initial condition over to the final condition

$$Q = A\,[U_2 - U_1] + A\int_{v_1}^{v_2} pdv.$$

In this equation the quantities $x$, $x'$, $x''$ are contained in the values for the energy, and indeed play a very important part therein; moreover, $x$ is implicitly contained in $v$, but notwithstanding this $\int_{v_1}^{v_2} pdv$ will be represented by the area included between the curved portion (more accurately, the projection on the $PV$ plane of the curve) representing the change of condition, the initial and the final ordinate and the portion of the axis of abscissas lying between these ordinates.

In the following sections the equations of condition for the individual stadia will now be considered, from them those of the characteristic curves (isotherms, adiabatics, and curves of constant quantities of saturation) will be deduced, and finally the course of these in the geometrical form of presentation will be investigated.

## A. THE DRY STAGE.

If we indicate by $p_\lambda$ the partial pressure exerted by the dry air, by $p_\delta$ the pressure resulting from the vapor and in general distinguish all quantities relating to the air and vapor, in an analogous manner by the same indices then we obtain directly

$$p = p_\lambda + p_\delta = \frac{R_\lambda\,T}{v} + x\,\frac{R_\delta\,T}{v}$$

or,

$$p\,v = (R_\lambda + x\,R_\delta)T \quad \ldots\ldots\ldots \quad (1)$$

* I adopt Zeuner's method of writing as more familiar to me; that is, I assume that the energy is expressed in units of work.

where $R_\lambda$ and $R_\delta$ are the constants of the equations of elasticity* for air and vapor, namely, $R_\lambda = 29.272$ and $R_\delta = 47.061$. If now $x$ remains constant, then for a constant $T$ this equation becomes that for an equilateral hyperbola. The isotherm for moist but not saturated air is therefore, as for dry air, an equilateral hyperbola or a portion of one; but for certain values of $v$ this equation loses its meaning.

The fundamental condition of the dry stage consists simply in this: that the pressure $p_\delta$ shall be smaller than the pressure $e$ corresponding to that of saturated vapor at the same temperature, or expressed algebraically

$$x\frac{R_\delta\,T}{v} < e,$$

where $e$ is a quantity depending upon $T$ and rapidly increasing with $T$.

The equation (1) holds good, however, only when

$$v > \frac{x\,R_\delta\,T}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, therefore, the effective portion of the hyperbola begins with the point whose abscissa is

$$v_s = \frac{X R_\delta\,T}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $e$ increases more rapidly than $T$, therefore this initial abscissa diminishes with increasing temperature. The initial points of all isotherms belonging to one and the same quantity of vapor $x$ lie therefore on a curve whose course is to be seen approximately on the figures to be given hereafter, in which such curves are designated by $S\,S$ with corresponding indices.

Hence the area within which are represented the conditions of the dry stadium for constant quantities of vapor $x$ is bounded by this curve on the side toward the co-ordinate axes. If this curve is so intersected by another curve, representing any change of condition that one passes from the side that is concave away from the axis over to its convex side then one leaves the dry stage and arrives in that of condensation, that is to say, in the rain or snow stage. I will therefore designate this curve as the curve of saturation or of dew-point. Points on this saturation curve are, in accord with the considerations just developed, determined by the hyperbolic isotherm and the initial abscissa. We can, however, equally well utilize also the corresponding ordinates and the initial abscissa. In the dry stage the following equation holds good for the ordinate:

$$p = p_\lambda + p_\delta < p_\lambda + e$$

consequently for the initial ordinate of the isotherm $T$, which will be

* Throughout this work, the "equation of elasticity" is used as a translation of the German *Zustandsgleichung*, as being preferable and more general than the ordinary expression "Equation for a gas" or "equation of condition."—*C. A.*

designated by $p_s$ as being located on the curve of saturation, the equation is

$$p_s = p_\lambda + e$$

or

$$p_s = \frac{R_\lambda\, T}{v_s} + e$$

or finally after substituting the value of $v_s$

$$p_s = \frac{R_\lambda + x\, R_\delta}{x R_\delta} . e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is therefore easy to determine the correlated values of $v_s$ and $p_s$ for any constant quantity of moisture $x$ and for any given temperature. On the other hand, only with the greatest difficulty and even then only by the use of empirical formulæ is it possible to bring the curve of saturation into the ordinary form:*

$$F\,(v_s,\, p_s) = 0.$$

We also will therefore entirely relinquish all attempts in this direction. By so much the more important is it therefore to show that from the curve of saturation for a given value of $x$ one can with ease deduce such curve for any other quantity of moisture. If $T$ and hence also $e$ is constant, then it directly follows from the equation

$$v_s = x \frac{R_\delta\, T}{e}$$

that the initial abscissas of isotherms corresponding to equal temperatures but different quantities of moisture are proportional to these quantities of moisture themselves, or if we indicate by $v_1$ and $v_2$ the initial abscissas belonging to the quantities of moisture $x_1$ and $x_2$, we have

$$v_1 : v_2 = x_1 : x_2.$$

If therefore we have any point such as $N_1$ of the dew-point curve $S_1$ corresponding to a given temperature $T$ this will be the initial point of the isotherm $(T, x_1)$ if as in the above given manner we indicate the point corresponding to the temperature $T$ and the quantity of vapor $x_1$; now draw the isotherm $(T, x_2)$ for the same temperature $T$ but for another quantity of vapor $x_2$, then we have only to increase or diminish the abscissa of $N_1$ in the ratio $x_2 : x_1$ in order to obtain the $x_2$ of the

* We see this from the following consideration: Since according to equation (4) $e = \varphi\,(p_s,\, x)$, and since again $e = F\,(T)$, and moreover $T = \psi\,(p_s,\, x)$; since further the equations (3) and (4) give $v_s\, p_s = (R_\lambda + x\, R_\delta)\, T$, therefore $v_s\, p_s = (R_\lambda + x\, R_\delta)\, .\ \psi\,(p_s,\, x)$, or if we omit $x$ from under the functional sign as being constant,

$$v_s\, p_s = (R_\lambda + x R_\delta) .\ \psi(p_s)$$

an equation which contains only $v_s$ and $p_s$, but not explicitly, as variables.

initial point $N_2$ of the isotherm $(Tx_2)$ originally considered as being unlimited; that is to say, in order to obtain a point in the dew-point curve $S_2$ corresponding to the quantity of moisture $x_2$.

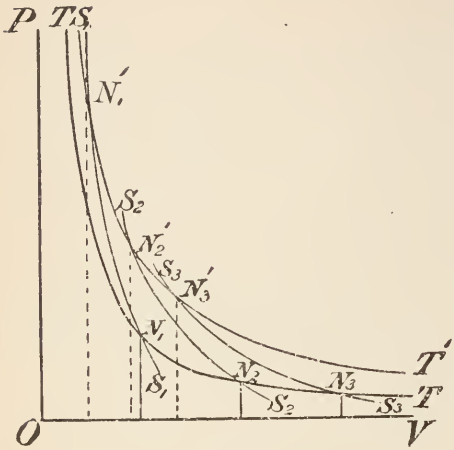



Fig. 29.

The dew-point curves $S_2$, $S_3$, of figure 29 therefore correspond respectively to quantities of vapor $x_2 = 2x_1$; $x_3 = 3x_1$ when $S_1$ corresponds to the quantity of vapor $x_1$.

The isotherms $(T, x_1)$ and $(T, x_2)$ run so near each other that they can only appear separated in a figure drawn to a very large scale,* since between the ordinates $p_1$ and $p_2$ of the two isotherms belonging to a given $v$, the following relations exist:

$$p_1 - p_2 = (x_1 - x_2)\,\frac{R_\delta\, T}{v}$$

or also

$$\frac{p_1}{p_2} = \frac{R_\lambda + x_1\, R_\delta}{R_\lambda + x_2\, R_\delta}$$

But this quotient is always very near unity, since all the values of $x$ that here come into consideration lie between zero and 0.03. In the majority of cases one can consider all the isotherms $(T, x)$ corresponding to a given value $T$ as coinciding with each other and have then only to remember that according to the value of $x$ they have their initial points at different places on the same hyperbola. Therefore from any one dew-point curve $S_1$ we obtain another one $S_2$ in that as already done in figure 29 we simply go with a constant ratio of expansion or compression further along an equilateral hyperbola drawn through $S_1$.

If we confine our consideration still to that portion of the plane of a constant quantity of vapor $x$ that lies to the right (that is to say, on that side of the dew-point curve that is distant from the coördinate axes) that is to say to the dry stage, then in this region the same theorems will hold good for the characteristic curves as for the so-called perfect gas, and particularly as for air, with such very small changes in the constants as depend on the mixing ratio [or the quantity $x$].

* It must here be expressly remarked that all the diagrams occuring in this memoir have a purely illustrative character. If we should introduce the separate quantities as they result from the computation the diagrams would lose perspicuity. The method here given therefore will need special modifications (as is hereafter to be shown) before it can be applied to graphical computations.

In this stage the isodynamic lines are also equilateral hyperbolas, and moreover the equation

$$pv^{\kappa}=p_1\, v_1{}^{\kappa}$$

holds good also for the adiabatic lines, when $p_1$ and $v_1$ relate to a definite initial condition, but $p$ and $v$ to an arbitrary final condition.

The constant $\varkappa$ can be adopted without notable error the same as for dry air, namely, $\varkappa = 1.41$. The quantity of vapor therefore disappears entirely from the formula and the adiabatics have the same course in all the planes corresponding to the different values of $x$. If now the adiabatic curves are considered as lines of constant entropy and we therefore take the equation $S–S_1 = 0$ as the fundamental condition where $- S$ is the entropy, then the equation of the adiabatic lines receives the following form

$$(c_p + xc_p{}^*) \log \frac{T}{T_1} - A\,(R_\lambda + x\,R_\delta)\log\frac{p}{p_1} = 0$$

where the capacity for heat of superheated aqueous vapor under constant pressure is indicated by $c_p{}^*$.

If one knows the path of any one adiabatic in the dry stage, then it is easy to construct any given number of others by means of it. To this end we consider that for any further progress along one and the same isotherm, according to well-known propositions, the following formula holds good for the quantity of heat needed in the expansion from $v_1$ to $v_2$:

$$Q_{1,\,2} = A\cdot R^*\ T \log\frac{v_2}{v_1}$$

where, for the sake of simplicity, we put $R_\lambda + xR_\delta = R^*$

Therefore we have

$$\frac{Q_{1,\,2}}{T} = A\ R^* \log\frac{v_2}{v_1}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But the quotient $\frac{Q_{1,\,2}}{T}$ is nothing else than the diminution of the entropy in the isothermal expansion from the volume $v_1$ to $v_2$. If, therefore, we start from a line of constant entropy (an adiabatic), and proceed along various isotherms that cut this curve, so that the ratio of expansion remains constant, then we attain to points on a second adiabatic.

If now we put $v_1{=}v$ and $v_2{=}v{+}\Delta v$, and then make $\Delta v{=}\nu v$, where $\nu$ is a constant (an appropriate proper fraction), and if in a corresponding manner we put $\Delta Q$ for $Q$ and $\Delta S$ for the difference of the entropy, we find

$$\Delta S = \frac{\Delta Q}{T} = A R^* \log\,(1 + \nu)$$

Therefore as soon as the course of one adiabatic line is known (just

* For the problems here presented, as is done by Zeuner in the application of the mechanical theory of heat to machines, it is recommended to give the positive sign to

as in the case of the dew-point curve) one can by a simple method of construction cover the plane of coördinates with a series of such adiabatics, each of which, with reference to its neighbor, shows a constant difference in the entropy by the amount $\Delta S$.

### B. THE RAIN STAGE.

For the rain stage, as already stated, there obtains the equation of mixture

$$M = 1 + x + x',$$

where $x'$ is in general very small, but $x$, except in exceptional cases, can only diminish. The equation of elasticity, on the other hand, is

$$p = \frac{R_\lambda T}{v} + e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $e$ is the vapor pressure, which in this stage, that is to say in the condition of saturation, depends simply and alone on the temperature $T$. Moreover, there obtains also the equation developed as a limiting condition in Art. 3 above, viz:

$$e = \frac{x R_\delta T}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This last formula shows at once the above suggested fact, that here we have in general to do with changes that are reversible to only a very limited extent. If, for instance, $T$ is put constant while $v$ increases, then the equation can only be fulfilled when $x$ increases. This same holds good (because $e$ increases rapidly with increasing $T$) when $v$ is kept constant and $T$ increases, or, as expressed still more generally, it holds good for all changes in condition that are represented in the diagram by a movement toward the concave side of the dew-point curve.

But an increase of $x$ is only to a very limited extent possible in general in the free atmosphere, namely, only when liquid water, in addition to the vapor, is suspended in the air, and only so long as this store of liquid holds out. The latter in most cases is soon exhausted, since it is precisely the liquid drops of water that fall as rain as soon as their mass becomes considerable.

Therefore in the rain-stage, changes of condition toward the concave side of the dew-point curve are possible only to a very limited extent and only until the condition of supersaturation comes to its end and

---

the quantity of heat communicated to the air. Therefore an increase of the quotient $\frac{Q}{T}$ corresponds to a diminution of the entropy according to the definition of entropy as given by Clausius (see Clausius's Collected Memoirs, Brunswick, 1884) Memoir IV, page 140, and Memoir VI, page 276.

becomes that of simple saturation.* This occurs as soon as the curve of change of condition attains the dew-point curve $x + x'$. Having in mind the geometrical presentation one can express this proposition as follows:

In the rain or snow stage, changes of condition are only reversible when and so long as they find their representation above the dew-point surface. If they find this in the dew-point surface itself, then only those changes are possible by which the representative point approaches the quasi horizontal coördinate plane, that is to say slides down toward the surface or in the limiting case becomes the dew-point curve itself. An ascent to the dew-point surface is in the free atmosphere only imaginable in exceptional cases (as for instance in case of the falling of rain through other layers or the mixing of other layers with moist air); a further progress toward the concave side of the dew-point curve or toward the lower side of the dew-point surface indicates a transition over into the dry stage.

Therefore in making use of the graphic presentation one must always keep in mind that in the rain and snow stages the curves in general can only be travelled over in one direction best represented by arrows and that a backward movement on the same curve is an impossibility. Nevertheless for the forward progress in the one possible direction exactly the same formulæ are applicable as for the reversible changes of condition. Therefore the case here occurring may with propriety be designated as "limited reversible."

We now turn to the consideration of the isotherm and the adiabatic for the rain stage. The equation of the isotherm we obtain at once as soon as we consider the temperature $T$ as constant in the equation of elasticity

$$p = \frac{R_\lambda T}{v} + e.$$

Since in this case $e$ is also constant, therefore this curve as in the dry stage is an equilateral hyperbola, one of whose asymptotes, as in the dry stadium, coincides with the axis of $p$, but the other is by the small quantity $e$ shoved from the axis of $v$ toward the side of positive $p$. At the same time however, in so far as we exclude super-saturation and starting from a given initial condition, this equation holds good only for diminishing values of $v$.

Moreover a glance at the equations of the isotherms in the dry and the rain stages suffices to show us that the two curves for any given temperature differ from each other only very little and that in the transition from the dry to the rain stage only a very small indentation

* In a certain sense the case where liquid water or ice is mixed with the air should certainly also be called that of super saturation, but of course with the reservation that any confusion with the condition of super-saturation properly so called, in which the excess above the quantity needed for saturation is present in gaseous form, shall be excluded.

can be seen with the vextex toward the right and above. This results from the circumstance that the isotherm for the rain stage contains the initial points, of all isotherms for the dry stage, which points correspond to values of $x_a$ that are smaller than the value of $x_b$ from which one starts out.

In order to obtain the equation of the adiabatic we must know the quantity of heat, $dQ$, that is to be communicated for a very small change in the condition. This $dQ$ is composed of the quantity of heat $dQ_\lambda$ that is given to the dry air and of the quantity $dQ_\delta$ that is communicated to the intermingled water or aqueous vapor. The following equations hold good for these quantities:*

$$dQ_\lambda = C_v dT + AR_\lambda T\frac{dv}{v}$$

and $$dQ_\delta = Td\left(\frac{xr}{T}\right) + (x + x')\,dT$$

[Where $r$ is the quantity of heat required to vaporize a unit mass of water at the temperature $T$ and the pressure $p$.]

In these $x'$ has values that lie between 0 and $x_a - x$ where $x_a$ indicates the quantity of vapor that was given to the original kilogram in its passage from the dry stage to the rain stage. $x'$ is equal to 0 when all the condensed water immediately falls down and is thus separated from the mass; it is equal to $x_a - x$ when all such water is carried along with the mass. The two limiting cases will occur relatively quite seldom in nature, but since at present we have no basis for determining to what extent liquid water is suspended in the air or can be carried along with it, therefore one must in the theoretical investigation confine himself to these limiting cases. Expressed in the language of the graphic presentation one must content himself with investigating those cases in which the indicating point either remains in the same plane as in the dry stage or on the other hand goes further on over to the dew point surface itself. Hitherto the first case only has been taken into consideration, although in general the second better agrees with the conditions occurring in nature.

Therefore the above given equation for $dQ_\delta$ assumes different forms, according as we consider the one or the other limiting case and we have, either

$$dQ_\delta = Td\left(\frac{xr}{T}\right) + x_a dT$$

for the case where $x_a$ is constant when all the water formed by condensation remains suspended,

or $$dQ_\delta = Td\left(\frac{xr}{T}\right) + xdT$$

where $$x = \frac{ev}{R_\delta T}$$

for the case when all this water immediately separates from the mass.

* See Clausius Collected Memoirs, Brunswick, 1884, Memoir V, page 174, or Hirst's translation of Clausius, pages 153 and 353.

The first case corresponds to a super-saturation limited only by the original amount of water, or, as I will briefly call it, the "maximum super-saturation;" the second case corresponds to the "normal saturation," rejecting any supersaturation.

For the quantity of heat $dQ=dQ_\lambda+dQ_\delta$ communicated to the mixture we obtain therefore two equations, namely:

(1) For "maximum super-saturation:"

$$dQ=(c_v+x_a)dT+Td\left(\frac{xr}{T}\right)+AR_\lambda T\frac{dv}{v}. \quad . \quad . \quad . \quad . \quad (8)$$

(2) For the "normal saturation:"

$$dQ=c_v dT+x dT+Td\left(\frac{xr}{T}\right)+AR_\lambda T\frac{dv}{v}. \quad . \quad . \quad . \quad (9)$$

If we put $dQ=0$ then we obtain the differential equations of the adiabatics for the two limiting cases. But in doing this we ought not to overlook the fact that strictly speaking in satisfying the condition $dQ=0$ we have to do with an adiabatic in the ordinary sense of the word only in one of these limiting cases, namely, that of maximal supersaturation. For if we establish for the adiabatic the single condition that for the given change of condition heat shall be neither gained nor lost, then we have in both cases true adiabatics to deal with. If however we define the adiabatic change of condition as one in which not only all exterior work shall be done at the cost of the energy, but also where the whole loss of energy shall be consumed in exterior work then will the definition for the second limiting case and also for all intermediate cases corresponding to values of $x'>0$ and $x'<x_a-x'$ equally agree with changes of condition that satisfy the condition $dQ=0$.

When, namely, the condensed water separates from the mass the energy diminishes not only by the quantity needed for the performance of exterior work, but also further by that quantity which is carried away by the water that has precipitated at a given temperature. I will therefore call those changes of condition for which $dQ=0$ but $x+x' < x_a$, that is to say, those changes for which the water wholly or partly separates the "pseudo-adiabatic," and especially that curve which obtains for the complete discharge of the water of condensation, the "pseudo-adiabat."

Corresponding to this method of distinction the equation

$$(c_v+x_a)\, dT+Td\frac{xr}{T}+AR_\lambda\, T\frac{dv}{v}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

obtains for the adiabat and the equation

$$(c_v+x)\, dT+Td\left(\frac{xr}{T}\right)+AR_\lambda\, T\frac{dv}{v}=0 \quad . \quad . \quad . \quad . \quad (10)$$

obtains for the pseudo-adiabat.

From these two equations we see, first of all, that the pseudo-adiabat descends more rapidly than the adiabat. Since for $dv>0$ we always have $dT<0$ and since moreover $x<x_a$, therefore the absolute value of $dT$ in the case of pseudo-adiabatic expansion must be larger than for adiabatic; that is to say, the temperature must sink more rapidly when all the condensed water is immediately discharged than when it remains still suspended.

Furthermore, both curves must sink more rapidly than the dew-point curve, or, in other words, for $dv>0$ we must always have $dx<0$. This follows directly from the circumstance that in expansion along the dew-point curve heat is to be added as also is shown from the manner in which the adiabatics of the dry stage intersect this curve. On the other hand, changes of condition with increase of heat are always represented by curves that descend less rapidly toward the axis of abscissas than do the adiabatics.

Therefore in the expansion of air the adiabatics depart from the dew-point curve toward the axis of abscissas and therefore $x$ diminishes.

The equation (8) is easily integrated and thus gives the following equation of condition for the adiabat:

$$AR_\lambda \log\frac{v_2}{v_1}+(c_v+x_a)\log\frac{T_2}{T_1}+\frac{x_2r_2}{T_2}-\frac{x_1r_1}{T_1}=0 \quad . \quad . \quad . \quad . \quad (10)$$

or if $v$ is expressed in terms of $p$, $e$, and $T$ with the help of the equation of elasticity;

$$AR_\lambda \log\frac{p_1-e_1}{p_2-e_2}+(c_p+x_a)\log\frac{T_2}{T_1}+\frac{x_2r_2}{T_2}-\frac{x_1r_1}{T_1}=0 \quad . \quad . \quad . \quad (11)$$

or finally by consideration of equation (7) and by the substitution of the corresponding values of $x_1$ and $x_2$;

$$AR_\lambda \log\frac{v_2}{v_1}+(c_v+x_a)\log\frac{T_2}{T_1}+\frac{e_2v_2r_2}{R_\delta T_2^2}-\frac{e_1v_1r_1}{R_\delta T_1^2}=0 \quad . \quad . \quad (12)$$

or

$$AR_\lambda \log\frac{p_1-e_1}{p_2-e_2}+(c_p+x_a)\log\frac{T_2}{T_1}+\frac{R_\lambda}{R_\delta}\left[\frac{e_2r_2}{T_2(p_2-e_2)}-\frac{e_1r_1}{T_1(p_1-e_1)}\right]=0 \quad . \quad (13)$$

If we consider the final condition as variable and corresponding to this drop the subscript index 2 then the equations become the following:

$$AR_\lambda \log v+(c_v+x_a)\log T+\frac{xr}{T}=C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$-AR_\lambda \log(p-e)+(c_p+x_a)\log T+\frac{xr}{T}=C \quad . \quad . \quad . \quad . \quad (11a)$$

$$AR_\lambda \log v+(c_v+x_a)\log T+\frac{evr}{R_\delta T^2}=C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

$$-AR_\lambda \log(p-e)+(c_p+x_a)\log T+\frac{R_\lambda}{R_\delta}\cdot\frac{er}{T(p-e)}=C \quad . \quad . \quad (13a)$$

Simple as are these collected equations in certain respects, still none of them allow us to express the relation between $v$ and $T$ or $p$ and $T$ or even between $p$ and $v$ explicitly, and in using them we are obliged to proceed by trials.

On the other hand one can, in comparatively simple manner, construct the curves in question when we remember that the left-hand side of equations (10) to (13), in all cases, even when they are not equal to 0, must still always give the value of

$$\int_{(1)}^{(2)} \frac{dQ}{T}$$

when we take this integral from the initial condition $v_1 p_1$ to the final condition $v_2 p_2$, and thereby apply the notation of the limits as here given, and as is easily comprehended.

But this value is nothing else than the diminution of the entropy during the passage from the initial to the final condition.

If therefore we compute this quantity for various properly chosen pairs of $v_2$ and $p_2$ we thus obtain the value of the entropy for the corresponding points, excepting only a constant that holds good for the whole system. Thus we shall be enabled to interpolate the corresponding values for intermediate points and thus to draw lines of equal entropy, namely, adiabatics. It is especially desirable to so choose these points that they come to lie in regular succession on the isotherms.

Then we have for the difference of the entropy due to the passage from a point 1 to a point 2 of the same isotherm, that is to say, for $T_1 = T_2 = T$

$$\int_{(1)}^{(2)} \frac{dQ}{T} = \frac{Q_{1,2}}{T} = A\, R_\lambda \log \frac{v_2}{v_1} + (v_2 - v_1).\ \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $\tau = \dfrac{er}{R_\delta T^2}$ that is to say, a quantity that remains constant for the same isotherm. This equation also teaches that the isentropic curves in the rain stage cut the isotherms at more acute angles than in the dry stage, for which latter the equation (5) holds good, namely,

$$\frac{Q_{1,2}}{T} = AR^* \log \frac{v_2}{v_1}$$

From the comparison of both equations, (5) and (14), it follows that a given change of the entropy in the dry stage corresponds to a greater change of $v$ than in the rain stage. Since now the isotherms in both stages can be considered as having very nearly the same course and, when we consider a very small part of the coördinate plane, can be considered as parallel straight lines, therefore for the given change of entropy in the dry stage one has to go a greater distance along the isotherm than in the rain stage.

Since, however, on the other hand, the dew-point curves descend more rapidly than the isotherms toward the positive side of the axis of abscissas, therefore the adiabatics must experience a bend at the dew-point curve in the manner shown in the figure 30. In this $S\ S$ presents a part of a dew-point curve; $A\ A$, $A'\ A'$, etc., adiabatics; $T\ T$, $T'\ T'$, etc., isotherms.

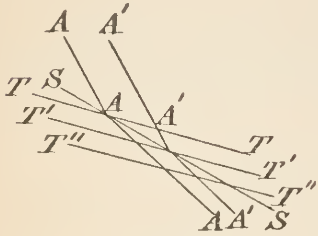



Fig. 30.

The differential equation of the pseudo-adiabatic can be treated in a similar manner to that of the adiabatic, but whereas in the adiabatic the integration was possible even when the connection of the independent variables was not explicitly given, on the other hand this is not the case for the pseudo-adiabatic. That is to say, instead of equation (10) we have for the pseudo-adiabatic the following:

$$A R_\lambda \log \frac{v_2}{v_1} + c_v \log \frac{T_2}{T_1} + \int_{(1)}^{(2)} \frac{x\,dT}{T} + \frac{x_2 r_2}{T_2} - \frac{x_1 r_1}{T_2} = 0,$$

or, preferably,

$$A R_\lambda \log \frac{v_2}{v_1} + (c_v + x_a) \log \frac{T_2}{T_1} - \int_{(1)}^{(2)} \frac{(x_a - x)\,dT}{T} + \frac{x_2 r_2}{T^2} - \frac{x_1 r_1}{T_1} = 0 \quad . \quad (15)$$

If therefore the point (1) is at once located in the dew-point curve then will $x_1 = x_a$; and if then we consider the point (2) alone as variable, that is to say, omit the subscript index 2 entirely, we obtain

$$A R_\lambda \log \frac{v}{v_1} + (c_v + x_a) \log \frac{T}{T_1} - \int_{(1)}^{(2)} \frac{(x_a - x)\,dT}{T} + \frac{xr}{T} - \frac{x_a r_1}{T_1} = 0 \quad . \quad . \ (16)$$

or after further modifications

$$A R_\lambda \log v + (c_v + x_a) \log T + \frac{xr}{T} - \int_{(1)}^{(2)} \frac{(x_a - x)\,dT}{T} = C \quad . \quad . \ (17)$$

We omit the development of formulæ entirely analogous to equations (11) etc., and it suffices to say that in them all the integral occurs as a correcting term. Happily its value remains always within very moderate limits, so that in the computation one can be satisfied with more or less perfect approximations. One can therefore omit the further consideration of the pseudo-adiabatic process and only call attention to the fact that it follows from equation (16) that the pseudo-adiabatic curve descends more rapidly than the adiabatic as was already pointed out above. For since when $v_2 > v_1$ we always have $dT < 0$ therefore the definite integral that still occurs in the equation has always a negative

value and because of the minus sign before the integral it therefore always exerts its influence in the same direction as the term $AR_\lambda \log \frac{v_2}{v_1}$. Therefore for the same starting point and for equal values of $T_2$, we must have $v_2$ in the case of the pseudo-adiabatic smaller than if we had gone along on the adiabatic.

## C. THE HAIL STAGE.

The above given equations hold good for the value $T > 273°$; as soon as the temperature 0° C. or the absolute temperature T=273 has been attained, then very different equations replace these but only when liquid water is present. In this last case the following equation of mixture holds good, namely:

$$M = 1 + x + x' + x'',$$

an equation that can only be true for the temperature 0° C. since only at this temperature can water and ice occur together. The equation of elasticity therefore then acquires the simple form

$$p = \frac{aR_\lambda}{v} + e_0$$

while the equation $x = \frac{ev}{R_\delta T}$ becomes $x = \frac{e_0 v}{aR_\delta}$ . . . (18)

wherein $a = 273$, $e_0 = 62.56$. But the one possible change of condition in this stage consists in an isothermic expansion. For this case therefore, the $dT$ also falls out of the equation for the transfer of heat and this takes the form,

$$dQ = r_0 dx - l dx'' + AR_\lambda a \frac{dv}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

[$r_0$=latent heat of evaporation at 0° C.; $l$=latent heat of liquefaction of ice.]

In this equation the first term on the right-hand side must be positive, the second must have a negative sign when $dx$ and $dx''$ are considered as positive, since an increase in the quantity of vapor $x$ makes an addition of heat necessary, but an increase in the formation of ice demands a withdrawal of heat.

If we put $dQ = 0$ then we have the differential equation of the adiabatic which in this case coincides with the isotherm and is moreover always a pseudo adiabat, since the ice that is formed falls away under all circumstances.

If we consider that

$$dx = \frac{e_0 dv}{aR_\delta}$$

then the differential equation of the adiabat takes the form

$$AR_\lambda a \frac{dv}{v} + \frac{r_0 e_0}{aR_\delta} dv - l dx'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

hence we obtain by integration

$$AR_\lambda a \log\frac{v_2}{v_1}+\frac{r_0 e_0}{aR_\delta}(v_2-v_1)-lx_2''=0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where we assume the integral to be taken throughout the whole stage from the initial value $v_1$ that corresponds the entrance into this stage to the final value $v_2$ that refers to the exit therefrom, and remember that the initial value of $x''$ namely, $x_1''$ is equal to 0 under these conditions. If however the integral extends only up to a value of $v$ lying between these two limits and which $v$ can then be considered as variable, then the equation can be again brought into a form analogous to that above given and we obtain

$$AR_\lambda a \log v+\frac{r_0 e_0}{aR_\delta}v-lx''=C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This equation allows us to see directly that for increasing values of $v$ that is to say for continued progressive expansion the quantity of hail also steadily increases whereas on the other hand from [equation (18) or] the expression

$$dx=\frac{e_0}{aR_\delta}dv$$

it follows that an evaporation goes hand in hand with the freezing of the water, so that at the end of the hail stage the quantity of vapor present is greater than it was at the entrance upon this stage.

With the help of the above described geometrical presentation we represent these results in the following manner.

The condition that must exist at the entrance upon the hail stage finds its representation at the termination $N'$ of a straight line $N_0N'$ perpendicular to the chief plane of coördinates and which rises up above the dew point surface. The length of this straight line is $x+x'$. It cuts the dew-point surface at a point $N$ that is distant from the plane of $PV$ by the quantity $x$. If now the mixture expands along the isotherm then $N$ rises along the dew-point surface slowly upwards, while the foot $N_0$ of the straight line advances along an equilateral hyperbola. But at the same time, the total quantity $x+x'$ diminishes in consequence of the discharge of the ice and $N'$ sinks correspondingly down until $N$ and $N'$ coincide in a single point $N_2$ and with this the hail stage has reached its end.

It is now of especial importance to learn how much water is thrown down in the form of hail; this question is answered by the following consideration. At the beginning of this stage we have only water and vapor, at the end only ice and vapor while the sum of these in the first and in the second case remain the same, if we take the precipitated ice also into the computation. Let $x_1'$ be the quantity of liquid water present

at the entrance into the hail stage, then according to what has just been said,

$$x'_1+x_1=x''_2+x_2$$

or

$$x''_2=x'_1-(x_2-x_1)$$

or finally, making use of the equation (18),

$$x''_2=x'_1-\frac{e_0}{aR_\delta}(v_2-v_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If we substitute this value in equation (21) then after an easy transformation we find

$$AR_\lambda a \log \frac{v_2}{v_1}+\frac{(r_0+l)e_0}{aR_\delta}(v_2-v_1)=lx'_1 \quad . \quad . \quad . \quad . \quad (24)$$

From this we can now first find $v_2$ by trial; the value thus found can be substituted in equation (23), whence in this manner $x''_2$ is found.

If we are justified in the assumption that all the vapor of water originally present is also after the condensation carried along until the entrance upon the hail stage, as appears to be the case in heavy hailstorms, then we have $x'_1=x_a$ and this is certainly large with respect to $x_1$ and $x_2$, and therefore so far as concerns the absolute value of $x''_2$ we may briefly put $x'_1=x''_2$, since the difference $x_2-x_1$ no longer comes into consideration. In cases in which this difference is appreciable, as for instance in the determination of $v_2$, one can of course not make use of the above approximation.

The equation (23) also shows in a very clear manner that in general the hail stage can only occur when liquid water is suspended in the air, that is to say, when $x'_1>0$ and that it acquires a greater extent the greater this value of $x'_1$, that is to say, the greater the quantity of suspended water that is present. Already, many years ago, Reye showed that on days of thunder storms the conditions are present in a conspicuous degree for the suspension and carrying up of water.

### D. THE SNOW STAGE.

If the air, saturated with aqueous vapor, be cooled below 0° C., then a part of this vapor must be precipitated as snow. The same formula can be applied to this process as that which we have used in the rain stage if only in place of the heat of evaporation $r$ there be inserted the sum $r+l$ where $l$ as above indicates the heat of liquefaction of ice. Therefore we can after small modifications apply to this stage all the equations developed in Section B. I confine myself to the re-writing in this modified form the two equations ($10a$) and (17); they thus become for the adiabatic

$$AR_\lambda \log v+(c_v+cx_c)\log T+\frac{x(r+l)}{T}=C \quad . \quad . \quad . \quad (25)$$

and for the pseudo-adiabatic

$$AR_\lambda \log v + (c_v + cx_c) \log T + \frac{x(r+l)}{T} - \int_a^T \frac{c(x_c - x)dT}{T} = C \quad . \quad . \quad (26)$$

where $x_c$ is the quantity of vapor at the beginning of the snow stage and the limits $a$ and $T$ are introduced into the integral, because in the hail stage, as in the beginning of the snow stage, $T=a=273$; $c$ is the specific heat of ice. Since $x$ is always smaller with diminishing $T$ and finally approximates to 0, therefore in the snow stage the deeper the temperature falls the more does the adiabatic approximate to that of the dry stage.

In the investigation just finished, attention has been especially directed to the course of the adiabatics, as had also been done in the above-mentioned older investigations. But in truth the adiabatic expansion and compression constitutes only a rare, exceptional case, as is already shown by the fact that the vertical temperature diminution computed under this assumption (according to the so-called convective equilibrium) results considerably larger than is given on the average by observations. It is therefore important to deduce the quantity of heat absorbed or emitted for given changes of condition, as determined by the values simultaneously observed of pressure, temperature, and moisture. In this process the method of geometrical presentation here developed is applied with great advantage. First, a glance at the manner in which the curve representing any given change of condition cuts the adiabatic suffices to give a decision as to whether in this change one has to do with a gain or loss of heat. Moreover the curve puts one in a position to deduce the quantity of heat exchanged by graphic planimetric methods or by a combination of computation with planimetric measures. According to what was said in the beginning the equation

$$Q = A[U_2 - U_1] + A\int_{(1)}^{(2)} pdv$$

holds good also for the processes here considered with three independent variables, and therefore also for a closed cyclic process

$$Q = AF,$$

where $F$ is the surface inclosed by the projection of the points that are imagined to be upon the $PV$ plane. Assuming that any change of condition is given by its projection on this plane and is represented by the line between the points $a$ and $b$ in Fig. 31, then we obtain the quantity of heat $Q_{a,b}$ involved in this change easily in the following manner: One draws through $a$ (Fig. 31) any curve of change of condition for which it may be easy to compute the increase or diminution of heat; also draw

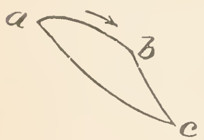


Fig. 31.

through $b$ an adiabatic and prolong both curves until they cut each other in a point, $c$; then is $Q_{b,c}=0$, and the quantity of heat is given by—

$$Q_{a,b}+Q_{c,a}=AF,$$

or,

$$Q_{a,b}-Q_{a,c}=AF,$$

and therefore, also,

$$Q_{a,b}=AF+Q_{a,c}.$$

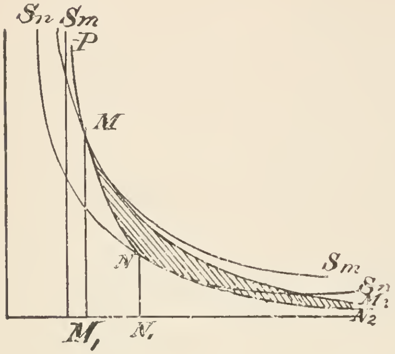
When now $Q_{a\,c}$ is determined by computation, but $F$ is found by planimetric method, this formula gives the value of $Q_{a,b}$.

If the curve $ac$ is the curve of constant energy (or isodynamic), then $Q_{a,c}=AL$, where $L$ is the exterior work and is therefore also directly obtained as a surface from the diagram, and then we have to execute the well-known graphic construction for the determination of the quantity of heat gained or lost by a given change of condition. But the method here given possesses the advantage of greater generality and much easier applicability.

This consideration also holds good when we have to do with limited reversible changes, only one has then to remember that the *closed* curve projected upon the plane of $PQ$ must also be the projection of a *closed* curve in space. If the curve in space that represents the change in condition is not closed, but if it only has the peculiarity that at the initial and final condition the coördinates $p$ and $v$ have equal values, then it indeed gives a closed projection, but the quantity of heat computed by the above-given method is erroneous, and that too by the quantity which corresponds to the increase in internal energy at the passage from the initial to the final point, that is to say, by the addition of the necessary quantity of vapor.

The circumstance that one and the same point of the $PV$ plane can correspond to very different conditions appears at first sight to exclude the general presentation of the processes in this plane alone, and thereby to materially diminish not only the applicability of the last-given construction but in general to detract from the whole conception here described. But by a closer consideration this is seen not to be the case; rather does it specially apply when for every point in the plane of $PV$ one has given the corresponding dew-point curve. An example will elucidate this: Let us assume that one desires to obtain an idea of the difference in the internal energy that is present in the dry stage for equal values of $p$ and $v$, but different quantities of vapor. If, in Fig. 32, $P$ is the point having the coördinates $p$ and $v$, but the quantity of vapor is in one case $x_m$ and in the other $x_n$, then these latter correspond to two different dew-point curves, $S_m$ and $S_n$. One can now convert the whole internal energy as it existed in the initial condition into external work by moving from the point $P$ forwards adiabatically to the absolute

zero point, which of course would demand a continuation of the adiabatic to infinity. If we do this in the case when the quantity of moisture is $x_m$, then will the projection of the adiabatic be represented by the line $PM\,M_2$, but by the line $PN\,N_2$ when the quantity of vapor is $x_n$, because in the first case under the pressure $M\,M_{,}$, in the second case under the pressure $N\,N_{,}$, the air passes out of the dry stage into the rain stage, and therefore the adiabatic descends according to another law, and in fact less precipitously. But the difference in the internal energy corresponding to the quantity of vapor belonging to the condition represented in $P$, and which by a self-evident notation is expressible as $A\,[U_m - U_n]$, is graphically represented by the surface $M_2\,M\,N\,N_2$, in so far as we imagine $M_2$ and $N_2$ extended to infinity and there united together.



g. 32.

When expressed analytically we find for this difference the expression—

$$A\,[U_m - U_n] = x_m\,t_m - x_n\,t_n + x_m\,\rho_m - x_n\,\rho_n,$$

wherein $\rho$ expresses the internal latent heat, and one has to remember that for given values of $p$ and $v$ the temperature varies with the intermixed aqueous vapor. However, this difference is so slight that in most cases it may be neglected, and one can therefore be satisfied with the approximation—

$$A\,[U_m - U_n] = (x_m - x_n)\,(t + \rho).$$

By this simplification the application of the above-described combination of planimetric measures and computations to the determination of the quantity of heat interchanged is very much lightened. If the temperatures are below 0° then the last formula must be slightly modified, which here need only to be referred to.

After having thus explained and established in general terms this new method of presenting the thermo-dynamic processes peculiar to the atmosphere their applicability will now be elucidated by a few examples.

(1) *The foehn.*

Moist air expands during its rise up the side of a mountain chain, and is then again compressed in its descent without having any heat added or withdrawn.

This is represented by a diagram, as shown in Fig. 33. Let $a$ be the initial condition, the corresponding dew-point curve $S_a$, then the air expands according to the adiabatic for the dry stadium until it cuts the curve $S_a$ in a point $b$, the curve $ab$ thus lies in a plane parallel to that of $PV$ distant therefrom by $x_a$. A glance at the course of the isotherms (of which only the one corresponding to the initial temperature is drawn and designated by $T_a$) shows that in this passage from $a$ over to $b$ the temperature sinks rapidly. As soon as the condition $b$ is reached the representative point [the indicator] slides down on the dew-point surface, the adiabatic of the dry stage goes over into $bc$, or that of the rain stage, and forms at $b$ an obtuse angle with the former. The temperature, with continued uniform progressive expansion, sinks much more slowly than before, water is condensed, since the curve $bc$ prolonged cuts the dew-point lines of lower quantities of vapor. The condensed water is deposited first as rain, afterwards as snow, and therefore $bc$ is the projection of the pseudo-adiabatic.

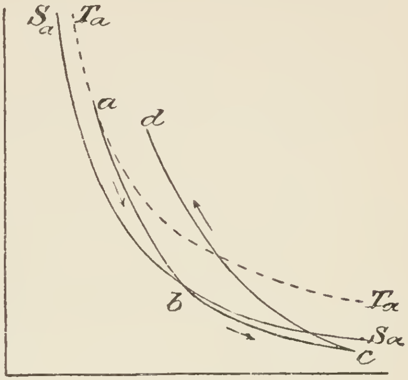


Fig. 33.

In this case the hail stage is entirely wanting, and although the cooling due to the continued expansion goes on beyond the freezing point, still this does not make itself so strongly felt in the course of the pseudo-adiabatic as that this transition should be perceptible in a drawing like the present diagram.

Let expansion continue up to a condition $c$, and now let compression occur, that is to say, the air reaches the summit or ridge of the divide and the ascent now becomes a descent on the other side. *Now*, all depends upon whether the condensed water was really completely precipitated or not. If not precipitated then during the compression there will be a retrogression of the indicator along the curve $bc$ in the direction from $c$ to $b$, and so much the farther along in proportion as more water has been carried with the air. If all the condensed water has remained suspended, then the change of condition in the retrograde direction continues back to $b$, and thence beyond to $a$, and we find on reaching the same level on the other side of the mountain again the same relations as in the beginning. This is always the case whenever the curve of saturation is not reached in the expansion, that is to say, when the whole process is entirely transacted in the dry stage in which case also the characteristic peculiarities of the *foehn* are wanting.

If however the rain stage is attained, and if in it the condensed water is actually precipitated then the process can not be reversed, and

then by the compression the change of condition from $c$ onward goes further along the adiabatic $cd$ of the dry stage. In this case a glance at the diagram shows immediately that for this change of condition the initial temperature will be attained even at a pressure that lies far below the initial pressure, and that in the farther progress towards pressures that are near the initial pressure, that is to say, in the descent to the old original level, much higher temperatures will be attained. At the same time the quantity of moisture is much less since the dew-point curve $S_c$ (which however is not drawn in order not to confuse the diagram), lies nearer the coördinate plane than the curve $S_a$, and since the curve of condition $cd$ remains with $S_c$ in the same plane which is parallel to the plane $PV$. The quantity of moisture which in the initial condition was $x_a$ is now at the end $x_d = x_c < x_a$, while for the temperature the equation $T_d > T_a$ holds good. Therefore after the passage over the mountain one has warm dry air, whereas at first it was cool and damp.

At the same time we see directly from the diagram that the characteristic peculiarities of the foehn must stand out so much the plainer in proportion as the point $a$ is nearer to the curve of saturation, that is to say, the warmer and moister the air is before its ascent and again, the longer the portion $b\ c$ is, that is to say, the more extensive is the expansion in the rain stage, or in other words, the higher the summit is that has to be surmounted.

Therefore we understand also at once why it is that in the Alps, independent of the prevailing conditions of atmospheric pressure, northerly foehns are so much rarer than the southerly foehns, as also why descending winds that have surmounted no summit, but have only passed along over a plateau, as for example the bora, have not the characteristic warmth of the foehn.

(2) *The interchange of air between cyclone and anti-cyclone in summer.*

Between an anti-cyclone and the cyclones that feed it, similar relations exist as between the masses of air on the two sides of a mountain range to be surmounted by them. In cyclones one has to do with an ascending current of air that afterwards descends in the anti-cyclone. Hence arises the precipitation in the region of the cyclone, the dryness and the clear sky in the region of anti-cyclone. But, whereas in the foehn the ascent and descent occur at points in the neighborhood of each other, so that in the short path there scarcely remains time for gain or loss of heat, but the whole process may in fact be considered as adiabatic; on the other hand very different relations obtain for the ascent and descent in cyclone and anti-cyclone. These two opposite processes in general occur at places so distant from each other that in the transit from one to the other extended opportunity is offered to take up or give out heat. In this process during the summer season the increase of heat prevails, but during the winter time the loss of heat; the day-time also

in its relations follows more or less closely the summer, while the night-time is like the winter.

Under the assumption of a prevailing increase of heat the process presents itself somewhat as shown in the diagram (Fig. 34); starting with the condition *a* (in a cyclonic area) the expansion with a diminution of temperature proceeds according to the curve *a b*, which descends rather less steeply than does the the adiabatic curve. Corresponding to this, and also without reference to the initial quantity of moisture, the dew-point curve is first attained later, that is to say, at a greater altitude above the earth's surface than it would be in adiabatic expansion.

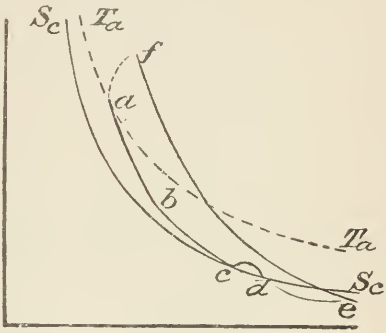



FIG. 34.

In the rain stage, therefore, the curve of change of condition experiences a deflection toward the upper side of the adiabatic, and therefore remains nearer the curve of saturation.

If now there occurs a still further greater addition of heat, as must be the case during the period of insolation and at great altitudes, where the condensation is less and the density of the clouds is correspondingly diminished, then the air can again pass over into the dry stage as is indicated in the portion *c d* of the curve.

Thus the upper limit of the first layer of clouds then would be at *c*. At this limit, during the summer days, more intense warming is in fact to be expected, which through a further expansion, that is to say at a greater altitude, on account of the diminished absorptive power of the atmosphere, again passes over into the approximate adiabatic *c d*, by which process, however, the dry stage is finally left and the snow stage *d e* is entered.

To this greater increase of heat at the upper limit of the clouds the fact is certainly to be ascribed that the cirrus (or snow) clouds are not directly continuous with the (lower or) water clouds, but generally separated from them through a wide space such as corresponds to the expansion from *c* to *d*.

During the descent in the anti-cyclone or by reason of the compression the process must take place according to the curve *e f*, which in general nearly agrees with the adiabatic of the dry stage. As we approach the earth's surface however, on account of the strong absorption of heat occurring there, then and for that reason this curve can depart to the right upwards from the adiabatic. This latter can however only occur temporarily, since in such a case we should have to do with a condition of unstable equilibrium.

The final pressure $p_f$, with which the sinking air reaches the ground in the anti-cyclone, is greater than the initial pressure $p_a$ that prevails at the ground within the cyclone, and correspondingly $f$ is higher above the axis of abscissas than $a$. In this case it may occur that the point $f$ comes to lie not only (as is self evident) above, but also to the right of $a$, so that $v_f > v_a$ or in other words that the air at the base of the anti-cyclone, notwithstanding the higher pressure, is specifically lighter than in the cyclone, because the temperature more than compensates for the influence of the pressure.

This shows in a very clear manner that in the exchange of air between cyclone and anticyclone we have to do not only with the specific weight of the mass of air, but that here dynamic relations are of first importance, a point to which Hann has called attention lately in the discussion of the observations taken on the Sonnblick.* It will be well in the more accurate investigation of this question to give increased attention to the processes above the aqueous clouds especially at their upper boundary surfaces.

As to the relations of the humidity to the processes just considered these are nearly the same as those in the case of the foehn. Here also, that is to say in the anticyclone, the air arrives in the neighborhood of the ground warm and dry, but in the immediate neighborhood of the ground the evaporation stimulated by unrestrained insolation will rapidly add moisture to the air, so that the indicator, which moving from $b$ nearly to $e$ has steadily approached the $PV$ plane and from $e$ on the way towards $f$ has remained a long time at the level of $e$, must now be imagined as rising immediately before reaching $f$. If now the air that has descended in an anticyclone again flows toward a new depression then will it (under the assumption of the same conditions in this as in the first cyclone), by reason of a continuous acquisition of aqueous vapor, pass through conditions that are represented in the diagram (Fig. 34) by the line $f\,a$. This line we have to imagine as slowly rising, so that the diagram here drawn presents in fact the projection of a closed line.

(3) *The interchange of air between cyclone and anti-cyclone in winter.*

In winter the diagram for this process of interchange has a figure essentially different from that in Summer. First, all changes in condition, at least insofar as concerns the initial and final conditions (see Fig. 35), take place nearer to the coördinate axes since the temperatures that come into consideration do not rise so high as in summer, and since, corresponding to this, the isotherms that lie far from the axis are not attained. Again, we have here lower pressure and higher temperature at the starting point $a$, but at the end $d$ higher pressure and lower temperature, so that $d$ is to be sought to the left and above $a$.

* *Meteorologische Zeitschrift*, 1888, vol. v, page 15.

Furthermore, the lines whose projections are here considered are not so far from the coördinate plane as in summer, because the absolute capacity for moisture remains always slight.

If now we follow more accurately the change of condition during ascent in the cyclone, we may at first assume that the process up to the attainment of the upper limit of the cloud stratum very nearly agrees with the adiabatic expansion, since below this limit radiation, either to or from, can only play an unimportant part. If however a departure from the adiabatic process does occur then it can be only in the opposite direction to that which occurs in summer, that is to say, the lines will descend more decidedly than in summer.

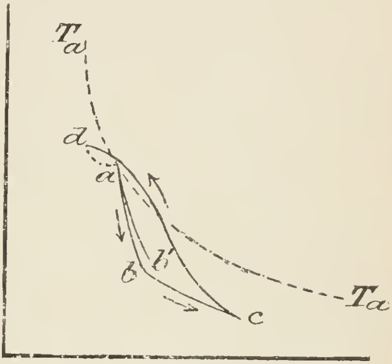


Fig. 35.

In Fig. 35 this latter case is assumed, as also that the passage out of the dry stage into the snow stage takes place immediately. From this point onwards the curve of condition again sinks more gradually, but with steadily increasing gradient in consequence of the overpowering cooling that certainly occurs at higher altitudes, until finally the turning point is attained and compression takes the place of expansion. The entire course of the change of condition to this point is presented by the curve *abc*. From this point onwards in consequence of the compression, the curve of condition must gradually advance to the point *d*. So far as our knowledge of the actual conditions of the atmosphere has attained hitherto, this gradual return to the point *d* occurs in such a way that at greater altitudes the compression proceeds adiabatically according to the adiabatic of the dry stage, whereas on approaching the ground the cooling by radiation that prevails there causes a deviation of the curve of condition from the adiabatic toward the axis of ordinates, and corresponding thereto the curve shows a course like *cd*. This curve however is nothing but the graphic expression for the well-known inversion that occurs on clear winter days in the vertical distribution of temperature. By reason of this inversion the curve near *d* approaches the dew-point curve, and can even pass it, so that condensation must occur and in the form of ground fog. But with the beginning of the formation of fog the radiation increases materially and corresponding to it the temperature diminution becomes always more intense with the proximity to the earth of the descending current of air.

Whether the passage from *c* to *d* be also possible by some other path by which from the very beginning of the compression the cooling and therewith the departure of the curve from the adiabatic makes itself felt, is a question that can be decided only after an accurate test com-

putation with the appropriate numerical data. At any rate such possible process would assume that in the anticyclone, at a certain height above the ground, exactly the same pressure and the same temperature prevail as at less altitudes above the base of the cyclone, since the projection of the curve of condition in this case must possess a double point.

These few examples, given only in their outlines, will suffice to enable one to realize the varied and useful applications that the method of graphic presentation here developed is capable of. By a further completion and development of the numerical side this method will give not only an excellent auxiliary means for the discussion and evaluation of existing data of observation, but above all will afford an indication as to the direction towards which material is to be collected in order to afford a deeper insight into the thermo-dynamics of the atmosphere.

If anything should seem especially suited to enable us to recognize the importance of the method of consideration here developed, it is the abundance of questions that press upon us at the first step we take in this way and that can at present be scarcely enumerated. I am thinking now, not only of the further development of theoretical consequences, therefore especially of the meaning of the thermal changes that occur in the atmosphere (especially the application of the second theorem of the mechanical theory of heat to these processes which may be developed in subsequent communications), but also, above all, of the stimuli that are to be derived therefrom to the observations of mountain stations, and especially in balloon voyages. For the latter it is full of meaning that in thermo-dynamic investigations the knowledge of the altitude above the sea can be entirely dispensed with and that it is entirely sufficient if we know the simultaneous values of the pressure, temperature, and moisture.

# XVI.

## ON THE THERMO-DYNAMICS OF THE ATMOSPHERE.*

### (*SECOND COMMUNICATION.*)

By Prof. Wilhelm von Bezold.

In a memoir published several months since,† I made an attempt to so extend the Clapeyron method of graphic presentation of thermodynamic processes as to allow of its application to atmospheric changes. At the same time I showed by some examples how with the assistance of this method of representation even complicated phenomena can be studied with comparative ease, and how by means of it we are put in the position of being able to draw most important conclusions almost like child's play. In the following, the same method will be applied to other questions not then or only lightly touched upon.

First, I will treat of a conception that has lately been introduced into meteorology by von Helmholtz,‡ and which appears to me to possess great significance in this science. This is the idea of "wärmegehalt," or total amount of heat contained within a body. Helmholtz measures the heat contained in a mass of air by the absolute temperature that this same mass will assume when it is brought adiabatically to the normal pressure. The quantity that we here deal with is therefore not as one might easily have believed a quantity of heat but a temperature, and therefore it seemed to me, upon my first study of the memoir in question, desirable to replace the term "wärmegehalt" by another. In a conversation upon this matter von Helmholtz recognized the objection expressed by me as proper, and proposed that the word "wärmegehalt" should be replaced by the evidently much more proper expression "potential temperature." This latter expression will therefore be used exclusively in the following memoir, but at first this idea itself will be more accurately considered. Its presentation in a diagram will be attempted and a general theorem deduced from it.

---

* Read before the Academy of Sciences, Berlin, November 15, 1888. (Translated from the *Sitzungsberichte der König. Preuss. Akademie der Wissenschaften zu Berlin*, 1888, vol. XLVI, pp. 1189–1206.)

† [See the preceding number of this collection of Translations.]

‡ "On Movements in the Atmosphere," *Sitzb. Berlin Akad.*, 1888, vol. XLVI, p. 647. [See No. V of this collection of Translations.]

## I. THE POTENTIAL TEMPERATURE.

According to what has just been said the potential temperature is that absolute temperature that a body assumes when without gain or loss of heat it is adiabatically or pseudo-adiabatically reduced to the normal pressure. I intentionally give this definition the form here chosen since we are here concerned with the application of the idea to meteorological processes, and since in our case the processes without increase or loss of heat do not need to be strictly adiabatic in the ordinary sense of the word. As I have shown in the previous memoir we have only to do with adiabatic processes when the water formed by condensation does not fall to the earth but is carried along with the air, a condition that is only fulfilled in exceptional cases. As soon as water is lost, and this is generally the rule, even though no heat be gained or lost, we have to do with a process that is only pseudo adiabatic. When therefore in the following, mention is made of adiabatic changes, the pseudo-adiabatic will always be included therein in so far as this class is not excluded by the special term "strictly adiabatic."

This much being premised we may now first investigate whether and how the potential temperature can be represented in a diagram. The answer to this question is extremely simple. From the equation of condition for the dry stage

$$vp = R^* T$$

there results

$$v = \frac{R^*}{p} \cdot T$$

or if we substitute for $p$ the normal pressure $p_0$

$$v = \frac{R^*}{p_0} \cdot T.$$

Therefore under constant pressure the absolute temperature is simply proportional to the volume, that is to say to the abscissa. But this absolute temperature under the pressure $p_0$ is the "potential temperature" for all other conditions that find their representation on the adiabatic passing through the point whose coördinates are $v$ and $p_0$. We therefore obtain the following rule:

*If a condition is given that is represented in the diagram, Fig. 36, by the point $a$, then we find the corresponding potential temperature by drawing an adiabatic line through $a$ and seeking its point of intersection $N'$ with a straight line $P_0\ N$ drawn parallel to the axis of abscissas and distant therefrom by $p_0$. The distance of this point of intersection $N'$ from the axis of ordinates, namely, the abscissa of $N'$ (or $N'\ P_0$) is now a measure of the potential temperature.*

We find the numerical values of $v$ and $T$ belonging to $p_0$ (and which I will now designate by $v'$ and $T'$ corresponding to the point $N'$, while I

designate by $v_a$ and $T_a$ those corresponding to the initial condition $a$) by combining the equation of the adiabatic

$$p_a v^\kappa{}_a = p_0 v'^\kappa,$$

with the equation of elasticity

$$\frac{p_a v_a}{T_a} = \frac{p_0 v'}{T'} = R^*$$

and we thus obtain

$$T' = \left(\frac{p_0}{p_a}\right)^{\frac{\kappa-1}{\kappa}} T_a$$

where $\varkappa = 1.41$.*

But this simple method of consideration is only allowable so long as the changes of condition take place within the dry stage. If this stage is left then the potential temperature belonging to a definite intial point has no longer a constant value, but it increases with the quantity of precipitation that is lost. A glance at the figure suffices to show this:

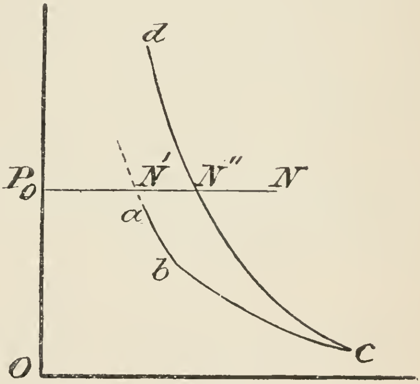


FIG. 36.

Assuming that the adiabatic of the dry stadium drawn through $a$ intersects the dew-point curve (which for simplicity is not shown in the figure) in $b$ and that we now allow the air to still further expand, then one has to pass from $b$ down along the adiabatic (or pseudo-adiabatic) of the rain or snow stage, that is to say along $bc$.

If now we seek the potential temperature for a point, $c$, of this line (in order to simplify the figure I have drawn the line $bc$ only just to this point), in that we bring it again adiabatically to the normal pressure, then one ought not to run back along the curve $bc$, since on account of the precipitated water the conditions represented by this portion of the line are not again attainable, but on the other hand one can only attain to the line of normal pressure by following the adiabatic $cd$ corresponding to the dry stage, but a dry stage with less quantity of aqueous vapor than before.

If we indicate by $N''$ the point at which this occurs or at which the normal pressure is thus attained, then as the measure of the potential

* In the previous memoir, in consequence of an oversight, $k$ was used instead of $\varkappa$ by von Bezold, but at his request this has been changed in the present translation.

temperature we have the length $P_0N''>P_0N'$; that is to say, the potential temperature $T''$, as attained by adiabatic change after passing into the condensation stage and after precipitation of some water, is higher than the potential temperature $T'$ of the initial condition and of all the conditions previously passed through in the dry stage 0.

Analytically this may be proved in the following manner:

For the transition from $a$ to $b$ the following equation obtains

$$p_a v_a^\kappa = p_b v_b^\kappa = C'$$

If this equation remains in force after crossing over the curve of saturation, then we obtain for the pressure proper to the volume $v_c$ a value $p_\gamma < p_c$ where $p_c$ is the pressure that in the condensation stage actually corresponds to the volume $v_c$.

But

$$p_c v_c^\kappa = p_d v_d^\kappa = C''$$

and since

$$p_\gamma v_c^\kappa = p_b v_b^\kappa = C'$$

and since also

$$p_\gamma < p_c$$

therefore

$$C' < C''.$$

But from this it further follows that $v''>v'$ and $T''>T'$ where $v'$ and $v''$ are the volumes corresponding to the normal pressure $p_0$ on the adiabatics $ab$ and $cd$; hence,

$$p_0 v'^\kappa = C'$$

and

$$p_0 v''^\kappa = C'',$$

beside which the following equation holds good:

$$v' : v'' = T' : T''.$$

Thus we attain to the theorem

*In adiabatic changes of condition in moist air the potential temperature remains unchanged so long as the dry stage continues, but it rises with the occurrence of condensation and so much the more in proportion as more water is discharged.*

Since in the free atmosphere, in general, evaporation does not occur and since also the carrying along of all the water that is formed, at least in the case of heavy condensation, must be considered as an exceptional case only, therefore, this theorem can also be brought into the following form:

*Adiabatic changes of condition in the free atmosphere, assuming that there is no evaporation, either leave the potential temperature unchanged or elevate it.*

From this theorem, which in its latter form reminds one of the theorem of Clausius in respect to the entropy, "The entropy strives towards a maximum," though not identical with it, one can draw consequences of the greatest importance. The next two sections will be devoted to these.

## II. THE VERTICAL TEMPERATURE GRADIENT.

All motions in the atmosphere can be considered as analyzed into vertical and horizontal components. The latter, in so far as they do not closely follow the irregularities of the earth's surface, are subject in only a slight degree to thermo-dynamic changes. On the other hand, in consequence of the expansion or compression in ascending and descending currents, the thermo-dynamic cooling or warming plays a very important rôle. The horizontal movements will therefore for the present be left entirely out of consideration, but the processes going on in the vertical currents will be thoroughly investigated. The changes of condition going on within ascending and descending currents must be considered in the free atmosphere as adiabatic so long as we content ourselves with a first approximation, and that we must do at first, since in the free air there is only a small opportunity given for active radiation and absorption. On the other hand the increase and diminution of heat will always make themselves felt decisively either where the absorbtivity and emissivity are remarkably increased or where the air comes into direct contact with bodies which themselves can strongly emit and absorb or otherwise take in or give out heat. This is the case:

(*a*) In the neighborhood of the earth's surface, where besides the increase in absorbtivity and emissivity of the air due to cloud or fog, the warming and cooling of the ground by radiation, as well as the evaporation, the formation of dew or frost, the thawing and freezing, have a powerful influence.

(*b*) In fog or cloud, which also possess a special power of absorbtion and emission, and where moreover evaporation can occur; and especially is it the upper limiting layer of clouds that one has to take into consideration.

In so far therefore as one can leave out of consideration the special localities just indicated, as also the mixture with other masses of air, one can approximately consider the processes in ascending and descending air currents as adiabatic. Even taking into consideration the special locations above mentioned, one can consider a scheme drawn up under the assumption of adiabatic change as to a certain extent an average or normal scheme, since such a scheme always occupies an intermediate position between those where the incoming radiation and those where the outgoing radiation prevails. How such a prevalence of either radiation must show itself has already been indicated in the previous communication [*p.* 212], where the interchange of air

between cyclone and anticyclone in summer and winter was investigated, at least in its principal features.

But in this study it is not necessary to limit oneself to the summer or the winter, but rather one can apply the scheme for the summer generally to all cases where the radiation is in excess, that is to say, not only to the summer time in general, but to the day-time and the hot zone; the scheme for the winter, on the other hand, is applicable not only to the winter season, but to the night-time and the cold zones of the earth. This normal scheme for the ascending and descending currents will therefore appear as shown in Fig. 36. The portion *a b* has reference to the ascending current in the dry stage, *b c* is its continuation in the condensation stage, finally *c d* is the portion of the curve that corresponds to the descending current.

This scheme differs only a little from that communicated in the first memoir. (For the case of the foehn, see page 240.) We can not expect it to be otherwise, since in the foehn one has also to do with an ascending and descending current of air in which the velocity with which the whole process goes on affords only a small opportunity for the gain and loss of heat. However, the diagram given in figure 36 as the "normal scheme" differs from that which obtains for the foehn in this respect, that the branch *cd* is longer. This is due to the fact that in the ordinary interchange between cyclone and anticyclone there always prevails a higher pressure at the base of the latter than at the base of the former; that is to say, the ending point *d* in the normal scheme must always lie higher than the starting point *a*, which is not the case in the foehn diagram. In general, one has to consider the process in the foehn as only a feature inserted into the normal interchange between anticyclone and cyclone. In the foehn the passage over the mountain chain forces the air in its normal interchange to describe an antecedent ascent and a subsequent descent which is only then followed by the definitive ascent in the cyclone. This being premised, the processes in the interchange, according to the normal scheme, will now be more precisely considered.

If we introduce the conception of the potential temperature, we attain the following theorems without any difficulty:

(*a*) In the ascending branch* the potential temperature increases steadily from the beginning of the condensation; in the descending branch it remains constant at the maximum value attained in the whole process. This maximum value corresponds also to the highest point to which the air has risen in its path.

(*b*) The potential temperature of the upper strata of the atmosphere is in general higher than that of the lower.

The first of these two theorems results directly from the diagram; the second follows from the fact that in the lower stratum the potential

* By the ascending branch is meant the portion *ab* which corresponds to the ascent in the atmosphere; the portion *cd* is considered as the descending branch.

temperature must, in the continuous interchange between cyclone and anticyclone, retain an average value that lies between the maximum value $T''$ and the smaller value $T'$ corresponding to the base of the cyclone; that is to say, to the point $a$ on the diagram. This average value is, however, certainly smaller than the maximum value $T''$ corresponding to the highest point of the path, and therefore to the condition $c$, and thus the theorem ($b$) is proven. Hence it follows that in nature the diminution of temperature for a constant elevation, or we will rather say, for 100 metres, that is to say; the so-called vertical temperature gradient, is, in general, smaller than results from the theory of the dry stage. As is well known, this gradient is 0.993 for the latter stage, that is to say, under the assumption of adiabatic change one would expect in the dry stage a diminution of 1° centigrade in temperature for an ascent of 100 metres.

This value 0.993 I will call $\nu$.

The above given theorems concerning the potential temperature show at once that under the assumption of adiabatic exchange the real value of the temperature gradient must be less than $\nu$.

We reach this conclusion from the following considerations:

Let $t_a$ and $t_d$ be the temperatures at the bases of the cyclone and anticyclone respectively (that is to say, at the starting and resting point, of the ascending and descending currents) then, under the assumption of perfect adiabatic change, these will not greatly differ from the potential temperatures $T'$ and $T''$, as these correspond to the ascending and descending branches in the dry stadium, that is to say, to the conditions represented by the curved portions $ab$ and $cd$ in figure 36. In this process the departures from these temperatures are always of such a nature that $t_a < T'$ and $t_d > T''$. For, since the pressure $p_a$ at the base of the cyclone is certainly smaller than the normal pressure, but the pressure $p_b$ at the base of the anticyclone greater than it (at least when a normal pressure is chosen appropriate to this case, and therefore lying between $p_a$ and $p_b$), therefore the temperature $t_a$ is increased by referring it back to this pressure, while $t_d$ by the corresponding process is diminished. Since the statement is thus proven that $t_a < T'$ and $t_d > T''$, and since, moreover, $T'' > T'$, therefore, by so much the more must $t_d > t_a$.

At the highest point of its path, such as corresponds to the point $c$ of the diagram, the particle of air has a potential temperature $T''$ that is to say, precisely the same as at the end.

If now it be assumed that this point lies $100h$ metres above the earth's surface, then there results as temperature gradient for the descending branch that is to say, as the increase of temperature for each 100 metres of descent, the well-known value

$$n'' = \frac{t_d - t_c}{h} = \nu.$$

On the other hand, for the ascending branch we obtain a value

$$n' = \frac{t_a - t_c}{h},$$

if for the sake of simplicity the difference of temperature prevailing above and below be equally distributed throughout the whole height.

This simplification is, of course, not strictly correct since the ascending branch of the two stages certainly includes in itself several stages, *e. g.*, the dry stage, the rain or snow stage, and perhaps also the hail stage, or all together. Still the method of computation of the average gradient as given here in the formula is the only method that we can apply when we have only one upper and one lower station. The following considerations however remain applicable at least in a general way when we can apply more rigorous formula.

Namely, for purely adiabatic change in any case we have $t_a < t_d$, and therefore also

$$n' < n''.$$

We attain to the same result also when we simply consider that the vertical gradient within the condensation stage is materially smaller than in the dry stage. When, therefore, the greatest gradient coming into consideration in the ascending branch is $n'' = \nu$, then the average of all must certainly be smaller.

*Therefore, in purely adiabatic ascent and descent and passage into the condensation stage the mean vertical temperature gradient in the ascending branch is always smaller than in the descending.*

If now we imagine regions of ascending and descending currents alternately passing over one and the same point of the earth's surface, we thus obtain for the mean vertical temperature gradient above that point a value $n$ that certainly lies between $n'$ and $n''$ therefore satisfies the condition,

$$n' < n < n'',$$

wherein $n'' = \nu$ is nearly constant, $n'$ however varies within wide limits according to the initial temperature and the initial quantity of aqueous vapor contained in the air.

*Therefore, under the assumption of adiabatic changes, in moist air that reaches the point of condensation, the mean vertical temperature gradient is always smaller than in dry air.*

We see from this that the consideration of the condensation alone already suffices to explain at least the direction of the departure of the observed vertical temperature gradients from those computed under the assumption of dry air, even if we retain the assumption of purely adiabatic changes. But this latter assumption is in fact certainly never fulfilled exactly, and it is therefore necessary to examine more accurately the influence that the departure to one side or the other from this normal process may have upon the vertical temperature gradient.

This again is most simply done by the introduction of the idea of the potential temperature. We can, namely, bring together all of the considerations just expounded into the following theorems:

(1) *If the potential temperature above and below is the same* i. e., *constant throughout the whole layer of air under consideration, then the vertical temperature gradient has the well-known value* $n = \nu$.

(2) *If the potential temperature in the upper stratum is higher than in the lower stratum (and this is in general the case), then is the temperature gradient smaller, and smaller in proportion as for a given difference in altitude, the difference of the potential temperatures is larger.*

If we indicate the potential temperature of the upper stratum by $T_s$ and that of the lower stratum by $T_i$, then for $T_s > T_i$ we shall always have $n < \nu$ and in fact the differences $T_s - T_i$ and $\nu - n$ always increase simultaneously.

A decided cooling in the lowest stratum always causes a diminution of $T_i$ and with it also a diminution of $n$, whereby even a change in the sign of $n$ may occur within moderate altitudes. In the latter case, the temperature below is lower than in somewhat higher layers, and in that case we have the so called inversion of temperature. If the cooling is not sufficiently strong to bring about an actual inversion of the temperature, still it causes a diminution of the gradient. Such decided cooling always takes place in the lowest stratum at the time of increased radiation, therefore especially in the region of the anti-cyclone, *i. e.*, under a clear sky, in winter, and in the night-time. Therefore in the winter and in the night-time the vertical temperature gradient must be smaller than during the summer and day-time, even if inversion in the distribution of temperature does not occur. This result agrees perfectly with observations, as is especially proven by the many facts that Hann and others have collected from the Alpine regions.

On the other hand the investigation here carried out teaches that the inversion of temperature and the diminution of vertical gradient connected therewith are to be treated not as phenomena peculiar only to the mountain regions, but that we are to expect them also above the plains, and even above the ocean, at least insofar as the more violent movements of the air do not interfere therewith.

We are therefore obliged to agree with Woeikoff* when he from a few data draws the conclusion that this inversion is also to be expected in the region of the great winter anti-cyclone of eastern Siberia.

On the other hand I can not agree with him when he deduces from this the consequence that Messrs. Wild and Hann should have considered this circumstance in drawing their isotherms, and I consider the standpoint taken by them as perfectly justified.†

* Woeikoff, *Klimate der Erde*, German edition, 1887, Bd. II, p. 322; *Meteorologische Zeitschrift*, 1884, Bd. I, p. 443.

† Hann, *Atlas der Met.*, 1887, p. 5. Wild, *Repert.*, 1888, Bd. XI, Nr. 14.

A direct proof of the inversion of temperature above the lowlands can only be expected from balloon observations.

To what extent radiation causes the inversion or at least the diminution of the gradient we shall learn from a work now soon to be published, that Sühring* has executed at my recommendation, and in which the vertical gradients of temperature between the Eichberg and the Schneekoppe, as well as between Neuenburg and Chaumont, are investigated according to the separate percentages of cloudiness.

It is not improbable that also above the ocean, and even at the time of the stronger insolation, a diminution of gradient, if not even an inversion of temperature, occurs, since over the sea the rapid evaporation in connection with the mobility of the water puts an impassable limit to the rise of temperature. The stability of the Atlantic anticyclone during the summer months may be based upon this circumstance.

The cases in which an increase of heat occurs at the earth's surface need no special consideration in the questions here considered. The gradient can only for a short time exceed the value $\nu$, as determined for the expansion or compression of dry air. If this case occurs, then, according to the investigations of Reye and others, we have unstable equilibrium or a condition that can only exist temporarily, as in whirlwinds or thunderstorms. Therefore, even for the strongest insolation, the considerations above given continue to hold good.

On the other hand the fact must excite great consideration that, not only on the average of all cases, but also when we investigate only the region of ascending currents (and of these only those that are below the limit of clouds, that is to say, for moderate elevation of the upper station) we find that the vertical gradient is always decidedly smaller than $\nu$. The reason of this is principally to be sought in the fact that the above views as presented by me, as also by other investigators in this direction, all rest upon an implied assumption that is only allowable to a very limited extent. They are based namely upon the assumption that the air ascending from the earth experiences no change in its constitution, except that due to the loss of water consequent on the adiabatic expansion, *i.e.*, that it experiences no mixture with masses of air of other temperature or other degrees of moisture, as also that every particle of air considered in the interchange between cyclones and anti-cyclones describes the whole path from the earth's surface to the limit of the temperature and back again.

But this is by no means the case. Only a small fraction of the air under consideration actually comes in contact with or even in close proximity to the earth's surface; and similarly with the ascent to the limit of the atmosphere or at least to the highest stratum that at any time takes part in the process under consideration. Moreover in the

* Sühring, *Die vertikale Temperaturabnahme.* Inaugural Dissertation d. Universität, Berlin, 1890.

ascending whirl, masses of air are always drawn in from one side that had not yet sunk to the earth's surface and had remained correspondingly unaffected by the radiation and absorption that have their seat in that stratum, and which also had had no opportunity to take up water from the earth's surface. Since these masses of air coming from the upper portions of the anticyclone have in general higher potential and therefore also higher absolute temperature than the portions of the cyclone lying at equal altitudes above sea level, therefore the mixture of these will diminish the cooling of the ascending air and both thereby as also by reason of the lesser quantity of water that they possess, will delay the occurrence of condensation.

Therefore in the cyclone itself the vertical temperature gradient even beneath the clouds will not be so large as one would expect according to the law of the adiabatic changes for the dry stadium without mixture of foreign masses of air. Similar relations obtain, although not to an equally great extent, with regard to the descending current, which in its upper half is also fed by portions of the cyclone in which the condensation has not yet gone so far and has not yet attained the high potential temperature of the highest stratum concerned in the whole process. Therefore in reality both the ascending and the descending branches of the curve deviate from the schema of Figure 36, and in both of them the vertical gradient will more or less approximate the average as we find it when we consider the ascent and descent as a connected whole.

These considerations are entirely in accord with observed facts. Even when we deduce the vertical temperature gradient from observations at stations of which the upper one is not so high that it is frequently within the clouds, we attain to temperature gradients that in general are far less than that computed for the dry stage; this result is in great part only explicable as due to the above described mixture. The observations of the clouds also agree perfectly with what has been said, both with regard to the temperature conditions and the moisture.

Only the central part of the cyclone is to any considerable extent fed by masses of air that have flowed along the surface of the earth itself, as one can easily convince himself by a simple diagram;* whereas the periphery receives more and more air from the higher strata, whereby its lower boundary surface is raised but its power must be diminished. In fact also the clouds at the center of the cyclone hang down the lowest and are higher near the circumference, exactly as is demanded by the moisture conditions and the higher potential temperature of the intermixed masses of air. The fringe of clouds that we perceive beneath the layer of clouds that covers the sky (especially on wooded hills during the prevalence of a cyclone) and in which we can clearly follow the ascent of air in inclined paths, gives in connection with the

* See, for example, Mohn, *Grundzüge*, 3d edition, 1883, p. 251.

ragged clouds surrounding the border of the continuous cloud cover, an excellent picture of the mixture just described.

Of course it is understood that all these considerations relate only to the conditions that ordinarily occur in the interchange of air between cyclone and anti-cyclone.

Processes in which we have to do with unstable equilibrium (such as occur, for instance, in the great thunderstorms in front of an advancing current of air, where a whirl with a long horizontal axis rolls rapidly forward and brings simultaneously on the side of the descending current heavy rain-fall and great cooling with higher barometric pressure, while on the front or ascending side the cloudiness is just beginning)—such processes demand a very special investigation that may be postponed to some future occasion. At present only one more consequence will be drawn from the propositions relative to potential temperature which seems to me calculated to throw a new light on the interchange of heat in the atmosphere, and that especially demands consideration from a climatological point of view.

### III. ON COMPLEX CONVECTION.

It has been shown above that in the adiabatic transfer of air out of the cyclone into the anti-cyclone, the potential temperature in the descending branch is higher than in the ascending. Hence it follows that in the descending branch a higher temperature prevails after attaining the initial pressure than prevails at the initial point, and a still higher temperature prevails at the end of the descending branch, that is to say on the ground in the anti-cyclone where, according to experience as well as for mechanical reasons, the pressure is always higher. Therefore in this transfer of air we are concerned not only with a simple transfer of the quantity of heat belonging to the air at the base of the cyclone, which we can here temporarily call the original quantity of contained heat, but this quantity of heat is increased by that heat of condensation which in the condensation stage did a part of the work of expansion and thereby diminished the cooling to a smaller quantity than it otherwise would be.

Even when in consequence of the stronger abstraction of heat at the base of the anti-cyclone the air is finally colder than it would have been in purely adiabatic interchange; and even when temperature inversion has occurred, still the temperature at the end of the process is still always higher than if the transportation of the air had taken place at the level of the earth's surface and the cooling influences had remained the same.

*The heat of condensation or negative heat of evaporation, or as it was formerly called the liberated latent heat, accrues to the advantage of that region in which the descending current has reached the earth's surface.*

We can therefore compare the whole process with that of a steam heater.

Moist air rises in the cyclone, attains the condensation stage and cools from that time on less rapidly since the heat of condensation does a part of the necessary work. The heat thus saved then enters into the descending current and finally is carried to the point at which the descending current reaches the earth's surface.

I consider it proper to designate by a special word those transfers of heat in which, besides the transport of warm or cooled bodies, changes of the condition of aggregation also occur, and therefore propose the name "complex convection" or "complex transfer." Such complex convection is met with when vapor is formed at one place and precipitated at another, or when ice falls as snow or hail, or when it is transported in the form of icebergs by ocean currents. If we apply this designation to the above-given considerations we obtain the following proposition:

"*In consequence of complex convection the temperature in anti-cyclonal regions is always higher than would be the case in simple convection.*"

The application of this proposition to the warm zone is of very special interest (I designedly avoid saying Tropical Zone since I can not consider the warm zone as limited by the Tropics) that is to say to the calm zone and the rings of higher atmospheric pressure that border it on either side, of which rings however the northern one is frequently interrupted.

The proposition just enunciated teaches that these two rings in consequence of complex convection are much warmer than would be the case if in the whole interchange one had only to do with dry air or with movements on one level. The warm zone is therefore hereby broadened and at the same time there is found within it a diminution of the temperature gradients.

In the calm zone itself much heat is used in evaporation and hence, in connection with the diminution of insolation by the covering of clouds, as also by reason of the water precipitated from colder regions above, the rise of temperature above a given limit is prevented. The heat consumed by evaporation at the earth's surface or at the ocean's surface does its work at a greater altitude in the region of the clouds when liberated by the condensation, and thus diminishes the cooling of the ascending current only to again reappear below in both the belts of descending currents.

A further development of the climatological consequences deducible from these considerations does not belong here. But this much we see at once, that the conclusions drawn from the mechanical theory of heat without any hypothesis whatever stand in direct contradiction to the older meteorological views. Formerly it was taught that the descending trade wind by cooling delivers to higher latitudes the water brought with it from the calm zone. Similarly it was taught that the heat liberated during the condensation raised the temperature, and that this

higher temperature inured to the places at or above which the condensation occurred.

The mechanical theory of heat shows that the current ascending in the calm zone must precipitate its water right there in the form of tropical showers, and that then it must descend as a drier and also as a warmer current (except in so far as it does not experience any material cooling, especially at the earth's surface). This theory further shows that the heat of condensation, in so far as super-saturation proper does not come into consideration, never shows itself as actually warming but only as diminishing the cooling that accompanies the ascent of the air, so that the current arrives at the upper limit warmer than it would without the accompanying condensation, and that the heat thus economized benefits the point at which the descending current reaches the earth's surface.

The considerations here developed can of course only be considered as approximate steps that still await additions and corrections. To my eye they play a rôle similar to that of the investigation of the so-called solar climate in climatology. Moreover some of these have no claim to complete novelty, but will be found here and there in connection with other special investigations.

On the other hand, they have never as yet been developed in such general—and never in such a simple—manner as is here done with the help of the idea of "potential temperature" and of the theorems that it was possible to deduce from this as to the potential temperature of the different layers of air. The consequences that can be deduced from this as to the static relations of the atmosphere, especially with reference to the fundamentally different behavior of cyclones and anticyclones in winter and in summer, both in respect to their intensity and their duration, are delayed to a later communication.

[An Appendix as published in the original memoir by von Bezold is omitted from this translation, as it has been at the author's request incorporated in its proper place in the latter portion of his first communication.]

XVII.

# ON THE THERMO-DYNAMICS OF THE ATMOSPHERE.*

(*THIRD COMMUNICATION.*)

By Prof. WILHELM VON BEZOLD.

In the two papers previously published on the above subject the restrictive assumption has been always made that the masses of air under consideration experience no mixture with similar masses having other temperatures and other degrees of moisture. At the same time however it was shown that such mixtures must frequently occur in nature and that the investigations in question could possess only a restricted application so long as we neglect these processes.

For this reason therefore it is now necessary to extend the previous investigations in this direction.

But investigations on this subject have also a special interest because for a long time we formerly attributed too much importance to the mixing of masses of air of unequal temperature and near the point of saturation, whereas in more recent times we have gone to the opposite extreme and attributed to it scarcely any importance at all.

Following the example of James Hutton,† the mixture of such masses of air was, until within a few decades of years, considered as the principal cause of atmospheric precipitation.

Wettstein was (so far as I know) the first to antagonize this view‡ which however even to-day is still widely accepted.

He however fell into the opposite error in that he contended that, in general, precipitation never could occur by mixing.

Here, as in so many other points of modern meteorology, Hann§ first made the matter clear in that he, in the year 1874, proved that by mixture condensation could be indeed produced, but that the former method of computing the quantity of precipitation was affected by an error in principle after correcting for which the values obtained are so small

* Read before the Academy of Sciences at Berlin, October 17, 1889. [Translated from the *Sitzungsberichte der König. Preus. Akad. der Wissenschaften zu Berlin*, 1890, pp. 355–390.]

† *Roy. Soc. Edinb. Trans.*, 1788, Vol. I, pp. 41–86.

‡ *Vierteljahrss. d. naturf. Gesell. Zürich*, 1869, XIV, pp. 60–103.

§ *Ztschft. Oesterr. Gesell. Met.*, 1874, Vol. IX, pp. 292–296. [*Rep. Smithson.*, 1877, p. 385.]

that the production of a moderately heavy precipitation in this way is impossible.

At the same time he showed that the adiabatic expansion in this respect played an entirely different and much more important rôle, and that, in it we have to recognize the source of all considerable precipitations.

In this paper, so far as it concerned mixture Hann confined himself to the computation of an example from which it appeared that even under very improbable assumptions there could in this way only be realized very slight quantities of precipitations.

Pernter many years later* contributed to the solution of the problem in that he brought it into an exact mathematical form and at the same time also computed small numerical tables in order to facilitate the comprehension of the quantities that enter into the question.

But since the empiric formula for the tension of aqueous vapor enters into the expression given by Pernter, therefore the latter is rather complex and is not especially clear.

It seems therefore to me not only desirable but really necessary to take up the question anew and if possible prosecute it to a definite conclusion. This is the object of the following lines.

It will be shown how graphic methods give with extraordinary ease an insight into the whole theory of the mixture of air and how in such methods we possess at the same time the simplest means for the numerical evaluation of the quantities that enter into the question.

Various tables—some of which may also be welcome for other investigations—will also facilitate a general survey as well as the exact computations. After these preparatory sections there will be considered the various causes of the formation of precipitation, namely, direct cooling, adiabatic expansion, and mixture, in their relative importance and it will be shown how that only by the consideration of all these causes is it possible to obtain a deeper insight into the methods of the formation of clouds.

## (*a*.) THE MIXTURE OF QUANTITIES OF AIR OF UNEQUAL TEMPERATURE AND MOISTURE.

Before we proceed to the mathematical treatment of this problem we must first come to a clear understanding as to whether definite masses or definite volumes shall be made the basis of the computation.

At the first view it would seem appropriate to adopt the volume, since we can from well-known tables obtain directly the quantity of water which corresponds to the saturation of one unit of volume.

This is doubtless the reason why in the older investigations of this subject based on Hutton's theory, one always started with the consideration of the unit of volume, and why Hann—when he would

* *Zeitschft. Oesterr. Gesell. Met.*, 1882, Vol. XVII, pp. 421–426.

demonstrate the imperfections of this theory in his considerations on this subject, followed the earlier method of treatment, and adopted the volume as a basis.

This is also quite justifiable so far as concerns the first estimates, and I also recently have made the same application in a popular lecture.

But when one wishes to obtain exact formulæ this method brings him into difficulties. These arise from the fact that the capacity for heat of a unit of volume, the so called volume capacity, even without the consideration of the intermixed vapor of water, is to a high degree affected by pressure and temperature, so that no forms of approximation are allowable. The capacity for heat of the unit of mass of moist air, therefore its capacity for heat in the ordinary sense of the word, is entirely independent of the above mentioned quantities and is also so little influenced by the contained water within the limits that occur in meteorology that, as will later be more accurately shown, we can in the present question simply consider it as constant.

In order however not to lose the advantage that inures from the utilization of existing tables, I have computed for different pressures and successive degrees the quantity of aqueous vapor that is contained in a kilogram of saturated moist air for such pressures and temperatures as occur in the atmosphere and have communicated the table thus formed in an appendix to this paper (see page 287).

This table not only facilitates very considerably the solution of the questions that refer to the mixture of moist air, but it can also be applied with profit to many other investigations. After this preface the problem itself is to be considered more closely, and to this end an appropriate notation is first to be introduced.

Let there be

$m_1$ and $m_2$, the quantities expressed in kilograms, of air to be mixed together;

$t_1$ and $t_2$, their temperatures;

$y_1$ and $y_2$, the quantities expressed in grams, of vapor actually contained in a kilogram of moist air;

$y'_1$ and $y'_2$, the corresponding values of contained moisture in a kilogram of air at $t_1$ and $t_2$ in the saturated condition;

$R_1$ and $R_2$, the accompanying values in per cent. of the relative humidity.

$\rho_1$ and $\rho_2$, the same quantities expressed as fractions of unity, that is to say

$$\rho_1 = \frac{R_1}{100} \text{ and } \rho_2 = \frac{R_2}{100}.$$

$t_3$, $y_3$, $y'_3$, $R_3$, and $\rho_3$, the various values of the same above-named quantities in the mixture, in so far as the limit of saturation has not been exceeded, or at least no water has been lost, that is to say, true saturation exists.

$t$, $y$, $y'$, $R$, and $\rho$, the corresponding values after mixture and after the loss of the quantity of water that exceeds the normal quantity for saturation, or also, in general, any given group of the same quantities belonging together.

The pressure expressed in millimetres of mercury will as before be expressed by $\beta$; the maximum of the elastic force of the vapor will in a corresponding manner be expressed by $\varepsilon$. The pressure $\beta$ can be considered as constant during the process of mixing. This is allowable since, where mixture actually occurs, the two masses of air must necessarily exist under very nearly the same pressure and must also retain this [in the free air] even when on account of the mixing a change occurs in the total volume, which in general is very unimportant.

The problem of mixture becomes extremely simple so long as no precipitation of water occurs, that is to say so long as the quantities obtained by the mixture are to be indicated as in the above notation by the subscript 3.

In this case

$$y_3(m_1+m_2)=y_1m_1+y_2m_2$$

or

$$m_1(y_3-y_1)=m_2(y_2-y_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and further

$$c_1m_1(t_3-t_1)=m_2c_2(t_2-t_3)$$

where by $c_1$ and $c_2$ we understand the thermal capacities of the quantities of air to be mixed,* or since these quantities are to be considered equal

$$m_1(t_3-t_1)=m_2(t_2-t_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we combine the equations (1) and (2) we obtain (the mixing ratio)

$$\frac{y_3-y_1}{y_2-y_3}=\frac{t_3-t_1}{t_2-t_3}=\frac{m_2}{m_1}$$

which is the well known equation that holds good for the mixture of two quantities of the fluid in question, having two different temperatures.

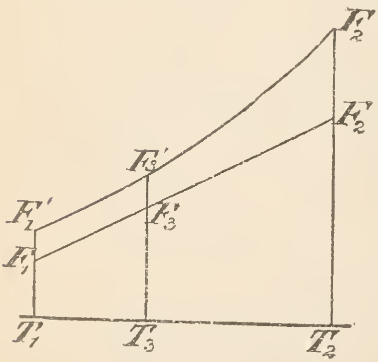


FIG. 37.

Since the graphic method will be chosen in the further development, therefore first of all this simple formula must be translated into a geometrical form.

To this end, in a rectangular system of coördinates, Fig. 37, the temperatures ($t$) are taken as abscissas, the quantities of moisture ($y$) as ordinates, and these are designated in the ordinary manner by $OT_1$, $OT_2$ . . . .

*Strictly speaking we should use mean values computed by a special formula between the above named $c_1$ and $c_2$ and that of the mixture $c_3$. Since, however, the values of $c$ scarcely differ from each other for the different temperatures and pressures, we can therefore omit this refinement.

$T_1F_1$, $T_2F_2$, etc.; in the figure the origin $O$ is omitted. We see at once that $F_3$ lies on the straight line drawn through $F_1$ and $F_2$ and that

$$\frac{T_1\,T_3}{T_2\,T_3}=\frac{T_3\,F_3-T_1\,F_1}{T_2\,F_2-T_3\,F_3}=\frac{m_2}{m_1}$$

In order now to obtain a decision as to the degree of saturation, we must also introduce into the diagram, as ordinates, along with the values of $y_1$, $y_2$, and $y_3$, also the values of $y_1'$, $y_2'$, and $y_3'$, corresponding to complete saturation. The ends of these ordinates, which are represented by $F_1'$, $F_2'$, and $F_3'$ in the diagram, all lie upon a curve that with increasing $t$ rises rapidly, and the equation* of which is

$$y=623\,\frac{\varepsilon}{\beta-0.377\,\varepsilon}$$

when for $\beta$ we insert the proper constant pressure.

With the assistance of this equation, or with the approximate formula obtained by development,

$$y=623\,\frac{\varepsilon}{\beta}+234.88\left(\frac{\varepsilon}{\beta}\right)^2$$

the tables communicated in the appendix [page 287] have been computed, by the help of which the curves can be easily constructed directly for the pressures therein considered, and which we can designate as curves of the quantity of vapor needed for saturation at the pressure $\beta$ [or for brevity, *the saturation curve*].

It will now suffice to cast a glance at the figure in order at once to obtain the following propositions:

(1) So long as for given temperatures $t_1$ and $t_2$, the values $\frac{y_1}{y_1'}=\rho_1$ and $\frac{y_2}{y_2'}=\rho_2$, remain within given limits, the straight line $F_1\,F_2$ passes entirely beneath the saturation curve, and therefore there can be no mixing-ratio for which condensation can occur.

(2) When $\rho_1$ and $\rho_2$ increase so much that the straight line $F_1\,F_2$ touches or cuts the saturation curve, as in figure (38), then there occurs either one or many mixing-ratios that may bring about condensation.

(3) When $R_1=R_2=100$, *i.e.*, when the two quantities of air to be mixed are saturated, then the straight line $F_1\,F_2$ coincides with the curve $F_1'\,F_2'$, and then for every mixture there occurs super-saturation or condensation.

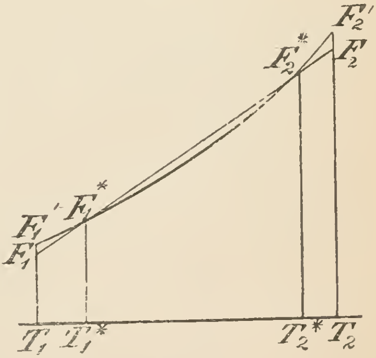



FIG. 38.

* Hann, *Zeit. Oesterr. Gesell. Met.*, 1874, vol. IX, p. 324. [Smithson Rep., 1877, p. 399.]

The investigation of the cases included in 2 can always be referred to case 3, since the points $F_1{}^*$ and $F_2{}^*$, in which the straight line $F_1\ F_2$ cuts the curve, $F_1'\ F_2'$ play precisely the same rôle in the second case as $F_1$ and $F_2$ in the third case.

If we consider more closely the propositions just enunciated, then we shall involuntarily be led to seek certain limiting values, the knowledge of which leads to the solution of the fundamental question whether, under given conditions, condensation will be possible or not.

The questions that obtrude in this connection are as follows:

(1) What limit must the relative humidity exceed for a given temperature of the components, or at least for one of them, in order that condensation may be possible for a properly chosen mixing-ratio?

(2) What limiting value must the relative humidity of one component exceed when the value of the other is given, and when also condensation is to become possible for a properly chosen mixing-ratio?

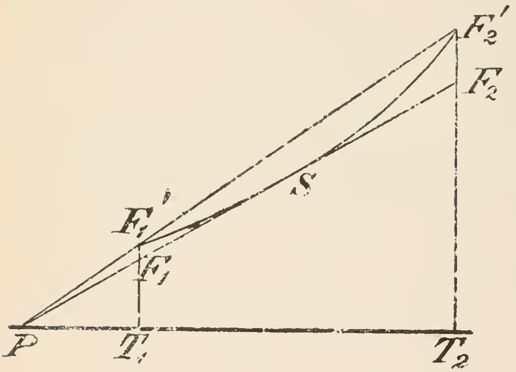



FIG. 39.

The first of these two questions can be expressed in the following form: When the limit of saturation is to be attained for an appropriate mixing ratio, and the relative humidity of both components is to be the same, what is the minimum value of this relative humidity?

That the knowledge of this minimum value is also a solution of question 1, we see most easily when we more accurately examine the answer to the question as last formulated.

We obtain this latter answer very easily through the following consideration: If $R_1$ is to equal $R_2$, then the straight line $F_1\ F_2$ must cut the axis of abscissæ at the same point $P$ Fig. 39, as does the prolongation of the chord $F_1'\ F_2'$. For if this condition is fulfilled then—

$$\frac{T_1\ F_1}{T_1\ F_1'} = \frac{T_2\ F_2}{T_2\ F_2'}$$

but now

$$\frac{T_1\ F_1}{T_1\ F_1'} = \frac{y_1}{y_1'} = \rho_1 = \frac{R_1}{100}$$

and

$$\frac{T_2\ F_2}{T_2\ F_2'} = \frac{y_2}{y_2'} = \rho_2 = \frac{R_2}{100}$$

and consequently, also

$$R_1 = R_2.$$

If now for a given value of $R_1=R_2$, which may be called $R_0$, the point of saturation is to be just attained by proper mixing, then the straight line $P\ F_1\ F_2$ must just touch the saturation curve $F_1'\ F_2'$.

The point of tangency $S$ gives therefore the temperature of the mixture for which saturation will be just attained, and hence also the mixing ratio.

But the value $R_0$, as the figure shows at the first glance, must be exceeded by at least one of the components when condensation is to become possible, and it therefore is precisely that limiting value that is desired in question No. (1).

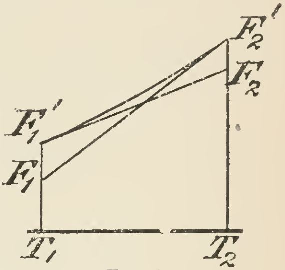


FIG. 40.

It is easily seen that the knowledge of these boundary values is of high importance, it is therefore carefully considered in tables to be subsequently communicated. Equally simple is the solution of the second question, which, however, will here be considered only under the special assumptions that $R_1$ or $R_2$ is equal to 100.

If $R_1=100$, that is to say, if the cooler of the two components is in the state of complete saturation, then we obtain the minimum value of $R_2$, when we, as in Fig. 40, draw at $F_1'$ a tangent to the saturation curve, and prolong this until it cuts the ordinate $F_2'\ T_2$ at the point $F_2$. The desired value is $R_2=100\,\frac{F_2 T_2}{F_2' T_2}$. As soon as $R_2$ exceeds this limit condensation occurs on mixing, provided that there is sufficient of the colder component, that is to say, provided $\frac{m_1}{m_2}$ is large enough.

If, however, we consider the other case as given and assume that $R_2=100$, that is to say, that the warmer component is saturated, then we find $R_1$ when at $T_2'$ we draw a tangent to the saturation curve and seek the intersection of it with the ordinate $F_1'\ T_1$.

Thus it becomes at once apparent to the eye that $R_1$ is always smaller than $R_2$, so that for sufficiently great distance between $T_1$ and $T_2$ the quantity $R_1$ can even attain a negative value, if such were imaginable.

The physical interpretation of this is that when warm saturated air is mixed with colder the latter can have a high degree of dryness and still condensation may occur for a proper mixing ratio; in many cases even the cooler air may be absolutely dry; it might even have a negative $R_1$ corresponding to its containing a certain mass of hygroscopic substance, if only there is sufficient quantity of warmer air, that is to say, if only $\frac{m_2}{m_1}$ is large enough.

In such cases, therefore, in place of the minimum value $R_1$ there occurs a limiting value of $\mu = \frac{m_2}{m_1}$, which must be exceeded if condensation is to occur.

These considerations show that mixtures of saturated warmer with unsaturated cooler air gives rise to condensations much more easily than do mixtures of saturated cooler with drier and warmer air.

The flow of a jet of saturated warmer air into a cool space must therefore be accompanied by much more powerful condensation than is the inflow of saturated colder air into a space filled with unsaturated warmer air.

The fact that clouds of vapor so easily arise over every open vessel filled with warm water, while the formation of fog near very cold bodies in warmer regions is much more rarely to be observed, gives an assurance of the correctness of this principle.

Whenever during moderately cool weather the door of a wash-house is opened great clouds of vapor pour out, but the opening of an ice cellar on a hot day has not a similar result.

Now that the limits have been determined within which, in general, mixture can occur, it is proper to give the quantity that can be precipitated by the condensation. Such precipitation occurs whenever the point $F_3$ lies above the saturation curve. For then the limit of saturation is exceeded, and by a quantity that is represented by the length $F_3 F_3' = y_3 - y_3'$.

This quantity, which will be designated by $a_3$, is that of which, before the writings of Wettstein and Hann, it was assumed that it was precipitated as water as the result of the mixing.

To what extent one was led into error by this assumption is most easily seen from the figure by the following considerations:

Let it be assumed that at first actual saturation occurs in the mixture, and let the whole quantity $y_3$ be actually présent in the form of vapor or aqueous gas, then will the gradual precipitation of the vapor be accompanied by a simultaneous warming.

The increase of temperature hereby brought about is found from the equation

$$1000\, c dt = - r dy,$$

where $c$ is the capacity for heat of the moist air under constant pressure, and $r$ is the latent heat of evaporation, and where $c$ is to be multiplied by 1,000, since we have taken a kilogram of the mixture, whereas $y$ is expressed in grams.

Since now, as will subsequently become evident, the temperature $t$ rises only a few degrees even for a very considerable supersaturation, therefore we can consider $\frac{c}{r}$ as constant in each individual case, and corresponding to this we obtain

$$y_3 - y = \frac{10^3 c}{r}(t - t_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $y$ and $t$ represent those values that are obtained after the precipitation of the water that is in excess of the limit of saturation.

In Fig. 41, therefore, we find this temperature $t$ in a very simple manner in that we draw through $F_3$ a straight line that makes with the axis of abscissas an angle

$$\alpha = \text{arc tang } \frac{10^3 c}{r}.$$

The point $F$, in which this straight line cuts the saturation curve, has the desired coördinates $t$ and $y$, whereas the quantity of precipitated water $a = y_3 - y$ is a quantity that is represented in the figure by the short line $F_3 i$. According to the old theory $t_3$ and $t$, as well as $y_3'$ and $y'$, or, what is the same, $y_3'$ and $y$, were considered respectively as the same. But now we see, as Hann had already shown in a special example, that this is not the case, but that $t > t_3$ and $y < y'_3$, and that correspondingly the actual quantity of water that can be precipitated in the most favorable case by mixing is

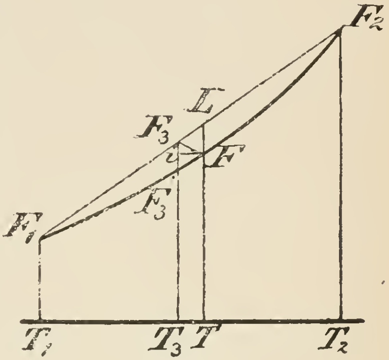


FIG. 41.

$$a = y_3 - y < a_3$$

that is to say less than any one has hitherto computed.

Since now $y = f(t)$ we can also put equation (3) in the form

$$t = t_3 + K(y_3 - f(t)),$$

where

$$K = \frac{r}{1000c} = \cot \alpha$$

and an empirical expression is to be substituted for $f(t)$.

This latter can, with the accuracy here desired, always be written under the form

$$f(t) = y_3 + A\,(t - t_3) + B\,(t - t_3)^2$$

so that we have only to consider the solution of an equation of the second degree.

However, the computation would be rather tedious and it is therefore decidedly preferable to execute this solution graphically, since this can be done rapidly and easily and preserves all the accuracy practically needed.

Of special importance is the circumstance that $K$ can, in general, be considered as a constant, to which only two different values have to be given, acording as it relates to values above or below 0°.

Hitherto we have implicitly assumed that the temperatures lay above 0°; if this is not the case, then, in place of $\frac{r}{c}$, the value $\frac{r+l}{c}$ is to be sub-

stituted where $l$ is the latent heat of melting ice. But so long as the melting point of ice is not exceeded, we can safely consider $K$ as constant, as the following consideration shows. The following equation,* the extremely simple deduction of which may here be omitted, gives the value of $c$:

$$c = 0.2375 + 0.00024\, y.$$

Now a glance at the table given in the appendix shows that $c$ will not exceed the value 0.2447, such as corresponds to a temperature 32° C. under 760 milimetres pressure. Since however on the other hand, for temperatures between 0° and 32° according to Regnault's figures,† $r$ is confined between the limits 606.5 and 584.2, therefore the extreme values that $\frac{r}{1000c}$ can have for a pressure of 760 milimetres are 2.55 for $t = 0°$ and 2.39 for $t = 32°$.

For lower pressures (that is to say at greater altitudes), $c$ is larger for a given temperature; but at the same time it is precisely under these conditions that only lower temperatures occur, and therefore only the higher values of $r$ are to be considered, so that $K$ still remains nearly within the same limits.

On account of the remarkably slight influence that the change of one unit in the first decimal place in the value of $K$ has on the final result we can for brevity put $K = 2.5$, so long as $t > 0°$.

If $t < 0°$, then we have to add the quantity 80 [calories] to the value of $r$. If we consider this and then compute $K$ for 0° and for $-30°$, first for $\beta = 760$ milimetres, and next for $\beta = 400$ milimetres we obtain as extreme values 2.87 and 2.98, so that here with even more right we can assume $K$ to be constant and as we in fact will do equal to 2.9.

According to this, without important error, we may consider the lines $F_3 F$, in general, as parallel straight lines which experience only a slight bend at the point corresponding to 0°.

In the actual application of the above-explained graphic method we do best to place upon the system of coördinates, on which we have entered the saturation curve, a group of straight lines representing the series $F_3 F$, of which those on the left of the zero coördinate are inclined to the axis of abscissas so that $\tan \alpha = \frac{1}{2.9}$, but those to the right of the zero coördinate have $\tan \alpha = \frac{1}{2.5}$.

---

* Hann, *Zeit. Oest. Gesell. Met.*, 1874, vol. IX, p. 324. [Smithson. Rep., 1877, p, 399.]

† According to the investigations of Dieterici (Wiedemann's *Annalen*, 1889, XXXVII, pp. 494–508), as well as according to those of Ekholm (*Bihang K. Svenska Vet. Akad. Handl.*, 1889, XV, Part I, No. 6.); these numbers are indeed not quite free from criticism. Since however on the one hand, the correction of these numbers scarcely comes into consideration in the final result here desired, and since on the other hand the value of the capacity for heat of dry air here adopted is based on the calorie used by Regnault, it appeared to me proper, if not even necessary, also to make use of the older value for $r$.

Special interest attends the question: In what ratio two quantities of air of given temperature and humidity must be mixed in order to obtain the greatest possible precipitation? The solution of this problem is given by a glance at Fig. 41. Since the quantity of precipitation is

$$a = F_3 F \sin \alpha,$$

therefore $a$ will be a maximum when $F_3 F$ has its greatest value. But this is evidently the case when the tangent at the point $F$ on the curve is parallel to the straight line $F_1 F_2$, or $F_1' F_2'$.

The point at which this tangent touches the curve can be determined either by construction and trial or, in case we have at hand a table of quantities of saturation, such as that in the appendix, computed for the barometric pressure in question, we have then to seek from it a value of $t$ such that

$$\frac{dy}{dt} = \frac{y_2 - y_1}{t_2 - t_1}$$

which is not difficult to do after constructing a corresponding supplementary table of differences for each tenth of a degree.

Having found the point $F$ we move further parallel to the previously mentioned group of straight lines until we strike the line $F_1 F_2$, and thus determine the point $F_3$, which on its part gives the point $T_3$, and thus the distances $T_1 T_3$ and $T_3 T_2$, whence results the mixing ratio that corresponds to the maximum precipitation. The precipitation itself we obtain from the above-given formula,

$$a = y_3 - y.$$

But we can also adopt another and purely numerical method for obtaining these quantities. For it is not difficult to see that $FL$ (Fig. 41) is also a maximum at the same time with $F_3 F$, where we designate by $L$ the point in which the prolongation of the ordinate $FT$ intersects the straight line $F_1 F_2$.

Moreover when we represent the line $FL$ by $l$, we have

$$\begin{aligned} l &= y_1 + (t - t_1) \tan \beta - y \\ &= t \tan \beta - y + y_1 - t_1 \tan \beta, \end{aligned}$$

where $\beta$ represents the angle that the line $F_1 F_2$ makes with the axis of abscissas, that is to say,

$$\tan \beta = \frac{y_2 - y_1}{t_2 - t_1}$$

Since the value of $y$ is not difficult to compute, when not taken directly from the table, one is therefore in condition to form a small auxiliary table for the value of the quantity $l$ for certain values of $t$, such as lie in the neighborhood of the one desired, and from it take out

the maximum value of $l$ or the value of $t$ corresponding thereto. Then the value of $a$ is given by the formula

$$a = l \frac{\tan \alpha}{\tan \alpha + \tan \beta}$$

whose deduction may here be omitted.

Thus both a numerical and a graphic method are at our disposal. If we follow the former, we can easily perceive that an extremely accurate knowledge of the quantity of vapor contained in a kilogram when in the condition of saturation is presupposed for an even moderately accurate determination of the value of $a$ and $t$, as well as of the ratio $\frac{m_1}{m_2}$.

Because of the unreliability of the data at hand the values obtained by computation have in themselves a rather high degree of uncertainty, so that one can equally well make use of the far more convenient graphic method without thereby in fact losing any thing in accuracy.

In this latter way the following small tables have been computed, which give the limiting cases above treated as especially interesting for the pressures 700 and 400 *mm.* and for temperatures that proceed by steps of 10° and thereby makes possible a quick review of the various questions relative to mixtures of air.

The first horizontal line of each of these twelve tables relates to the case where both component masses are completely saturated, and gives in the column $a$ the greatest precipitation that can occur* under these circumstances and under the most favorable mixing ratio $\frac{m_1}{m_2}$.

Therefore the $a$ on the first line of each table, gives the maximum possible precipitation that can be brought about by mixture at the given temperatures.

The second line of each table gives the value of the relative humidity which must (at least for one of the components) be exceeded if precipitation by mixture is to be any way possible. We also find on this line under the headings $t$ and $\frac{m_1}{m_2}$ the mixing temperature and the mixing ratio for which the point of saturation will be just attained when in both components the relative humidity has the minimum values, given under $R_1$ and $R_2$.

The third line shows the value of $R_2$ that must be exceeded by the relative humidity of the warmer component, if the cooler component is completely saturated and if precipitation is to become possible by mixture.

The fourth line gives the mixing ratio, which must be exceeded if precipitation is to become possible by means of any proper mixing

* [Expressed in grams of water per kilogram of moist air.]

ratio when the cooler component is perfectly dry and the warmer component perfectly saturated.

The fifth line shows, under $a$, the maximum precipitation that is conceivable under the last mentioned condition of the components as to humidity as well as the mixing ratio and mixing temperature at which this maximum precipitation is attainable.

In many cases no precipitation is possible with perfect dryness of the cooler component. In such cases the fourth line is the analogue of the third since it gives the minimum value which the relative humidity of the colder component must exceed if in general precipitation is to become possible by mixture. Under these conditions in the nature of the case the fifth line becomes a blank.

The tables as here given relate only to the two pressures 700 and 400 millimetres. Since however these include all altitudes between 680 and 5,150 metres, that is to say, those altitudes in which the formation of cloud or at least precipitation proper principally occurs, and since the supplementing of these tables by means of the table given in the appendix is not difficult, I have thought that I might confine myself to these special cases.

At any rate these will suffice in order to give a general orientation as to the quantities coming into consideration, and therefore the tables themselves are now given, and it need only be stated that the figures must be considered only as approximations, since in general they are based upon the first differential quotients, but occasionally on the second differential quotients of the curve of vapor-pressure, so that very small changes in the experimental data or in the method of interpolation must make themselves very sensible.

| $t_1$ | $t_2$ | $R_1$ | $R_2$ | $a$ | $t$ | $m_1 : m_2$ |
|---|---|---|---|---|---|---|
| | | $b=700^{mm}$; $t_2-t_1=20°$. | | | | |
| −20° | 0° | 100 | 100 | 0.4 | −9.0 | 102 : 98 |
| | | 76 | 76 | 1/∞ | −14.0 | 140 : 60 |
| | | 100 | 52 | 1/∞ | −20.0 | 1 : 0 |
| | | 0 | 100 | >0.0 | >−11.8 | <118 : 82 |
| | | 0 | 100 | <0.13 | ≷−5.5 | ≷60 : 140 |
| −10 | +10 | 100 | 100 | 0.55 | 1.0 | 106 : 94 |
| | | 81 | 81 | 1 ∞ | −2.8 | 128 : 72 |
| | | 100 | 61 | 1/∞ | −10.0 | 1 : 0 |
| | | 0 | 100 | >0.0 | >−0.1 | <1 : 1 |
| | | 0 | 100 | <0.2 | ≷0.5 | ≷54 : 146 |
| 0 | +20 | 100 | 100 | 0.75 | 11.9 | 108 : 92 |
| | | 86 | 86 | 1/∞ | 6.2 | 138 : 62 |
| | | 100 | 62 | 1/∞ | 0.0 | 1 : 0 |
| | | 0 | 100 | >0.0 | >12.2 | <80 : 120 |
| | | 0 | 100 | <0.2 | ≷16.7 | ≷37 : 163 |

| $t_1$ | $t_2$ | $R_1$ | $R_2$ | $a$ | $t$ | $m_1 : m_2$ |
|---|---|---|---|---|---|---|
| | | | $b = 700^{mm}$; $t_2 - t_1 = 10°$. | | | |
| −20 | −10 | 100 | 100 | 0.04 | −15.5 | 57 : 43 |
| | | 92 | 92 | 1/∞ | −16.0 | 60 : 40 |
| | | 100 | 82 | 1/∞ | −20.0 | 1 : 0 |
| | | 55 | 100 | 1/∞ | −10.0 | 0 : 1 |
| −10 | 0 | 100 | 100 | 0.11 | −4.0 | 43 : 57 |
| | | 94 | 94 | 1/∞ | −5.5 | 55 : 45 |
| | | 100 | 85 | 1/∞ | −10.0 | 1 : 0 |
| | | 47 | 100 | 1/∞ | 0.0 | 0 : 1 |
| 0 | +10 | 100 | 100 | 0.19 | 5.0 | 54 : 46 |
| | | 94 | 94 | 1/∞ | 4.5 | 55 : 45 |
| | | 100 | 87 | 1/∞ | 0.0 | 1 : 0 |
| | | 64 | 100 | 1/∞ | 10.0 | 0 : 1 |
| +10 | +20 | 100 | 100 | 0.21 | 14.5 | 55 : 45 |
| | | 94 | 94 | 1/∞ | 14.0 | 60 : 40 |
| | | 100 | 87 | 1/∞ | 10.0 | 1 : 0 |
| | | 76 | 100 | 1/∞ | 20.0 | 0 : 1 |
| | | | $b = 400^{mm}$; $t_2 - t_1 = 20°$. | | | |
| −20° | 0° | 100 | 100 | 0.50 | −9.5 | 108 : 92 |
| | | 76 | 76 | 1/∞ | −14.0 | 140 : 60 |
| | | 100 | 58 | 1/∞ | −20.0 | 1 : 0 |
| | | 0 | 100 | >1/∞ | >−11.8 | <118 : 82 |
| | | 0 | 100 | <0.2 | ≷−5.4 | ≷54 : 146 |
| −10 | +10 | 100 | 100 | 0.75 | 1.2 | 110 : 90 |
| | | 80 | 80 | 1/∞ | −3.3 | 133 : 67 |
| | | 100 | 65 | 1/∞ | −10.0 | 1 : 0 |
| | | 0 | 100 | >1/∞ | >0.3 | <97 : 103 |
| | | 0 | 100 | <0.2 | ≷6.0 | ≷45 : 155 |
| | | | $b = 400^{mm}$; $t_2 - t_1 = 10°$. | | | |
| −20 | −10 | 100 | 100 | 0.12 | −15.5 | 58 : 42 |
| | | 96 | 96 | 1/∞ | −16.0 | 60 : 40 |
| | | 100 | 85 | 1/∞ | −20.0 | 1 : 0 |
| | | 48 | 100 | 1/∞ | −10.0 | 0 : 1 |
| −10 | 0 | 100 | 100 | 0.17 | −4.5 | 50 : 50 |
| | | 94 | 94 | 1/∞ | −5.5 | 55 : 45 |
| | | 100 | 88 | 1/∞ | −10.0 | 1 : 0 |
| | | 52 | 100 | 1/∞ | −0.0 | 0 : 1 |
| 0 | 10 | 100 | 100 | 0.20 | 6.0 | 47 : 53 |
| | | 93 | 93 | 1/∞ | 5.0 | 50 : 50 |
| | | 100 | 86 | 1/∞ | 0.0 | 1 : 0 |
| | | 65 | 100 | 1/∞ | 10.0 | 0 : 1 |

In agreement with the previous results by Hann and Pernter, these tables show how small is the precipitation attainable by mixture when we consider components whose differences of temperature are even greater than ever occurs in nature.

Since on the other hand, according to the data recently collected by Hann,* quantities of water considerably greater than these can remain suspended in the air (as mist, fog, and cloud), therefore we see very plainly that, while the formation of cloud can be caused by mixture, yet the precipitation of rain or snow in any appreciable quantity can scarcely be brought about in this way.

At the same time the following diagram, which we here make use of for graphic computation, enables, in the most simple manner, to compare the quantity of precipitation formed by mixture with that which is produced by direct cooling as well as that produced by adiabatic expansion.

If we assume that by mixture under a favorable mixing ratio of saturated air at the temperature $t_2$ with other saturated air at the temperature $t_1$, the quantity of water $a$ is precipitated (see Fig. 42), then we obtain the same quantity of precipitation when we directly cool the component $y_2$, from its temperature $t_2$ to a new temperature $t_a$, for which we have $y'_a = y'_2 - a$, but $y'_a$ is the ordinate whose foot is $T_a$ in Fig. 42.

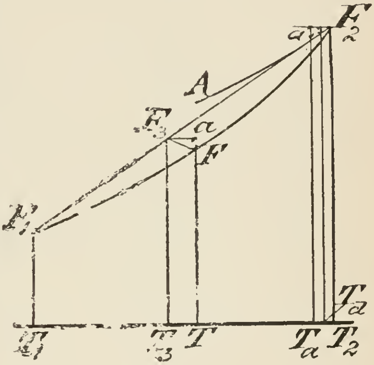


FIG. 42.

A glance at the general saturation curve suffices to show at once that the difference $t_2 - t_a$ is very much smaller than the difference $t_2 - t$; that is to say, that a very slight direct cooling affords as much precipitation as a considerable cooling by mixture with colder air, even when the latter is completely saturated.

The effect of adiabatic cooling is seen when in the diagram we draw the adiabatic curve as a function of the temperature and quantity of water contained in a kilogram of moist air.

Such an adiabatic curve sinks, as we easily perceive, rather more slowly from the right toward the left than the saturation curve. For since in this case the diminution of temperature goes hand in hand with the increase in volume, therefore, the quantity of moisture necessary for saturation will for falling temperatures be greater than it would be if the initial pressure were maintained; that is to say, than it would be by progressing along the saturation curve.

The adiabatic (which without any difficulty can be introduced into the diagram with sufficient accuracy with the aid of Hertz's Graphic Method*) will therefore have a path similar to that shown by the curve $F_2\,A$ in Fig. 42.

---

* *Meteorologische Zeitschrift*, 1889, vol. VI, pp. 303–306.

† *Meteorologische Zeitschrift*, 1884, vol. I, pl. VII. [See No. XIV of this collection of Translations.]

But in this case the lowering of the temperature must be forced down to $t_a$, if the quantity of precipitated water is to be equal to $a$, since then the equation

$$y'_2 - y'_a = a$$

holds good for $y'_a$, which represents the ordinate erected at $T_a$.

Here also the general course of the curve again shows that the fall of temperature necessary in order that a definite quantity of precipitation may be caused by adiabatic expansion is very much less than when the same quantity is to be produced by mixture.

A numerical example will best illustrate this principle: From the above-given small tables we see that at 700 millimetres pressure saturated air at 0° C. mixed with saturated air at 20° can precipitate at the most only 0.75 grams of water per kilogram of the mixture and that the final temperature of the mixture will be 11°.0; that is to say, for a cooling of the warmer component from 20° down to 11°.

By direct cooling, on the other hand, the same quantity of water would be precipitated from 1 kilogram of the warmer component when it is cooled from 20° down to 19°.2; whereas by adiabatic expansion a cooling of from 20° down to 18°.4 would be necessary; that is to say, a vertical ascent through a distance of about 310 metres.

This example shows in a very striking manner how slight need be the direct cooling by contact with cold objects, or by radiation, or even by adiabatic expansion, in order to produce quantities of precipitation, such as would by mixture be only obtainable in the extremest, scarcely imaginable cases.

With this the consideration of the mixture of masses of moist air may be brought to a close and only the single remark be made that the difference $t—t_3$ is smaller as the quantity $a$ of the precipitated liquid decreases. The amount of this difference will therefore only exceed the value of 1° or 2° in such extreme cases as are assumed in the previous tables and generally will remain far within this limit.

Therefore in the majority of cases the mixing temperature may, without important error, be put equal to that which we obtain by mixing equal masses of dry air, whereby many computations experience a great simplification.

### (*b*.) SUPER SATURATED AIR.

In the foregoing solution of the problem of mixture it was assumed for the sake of simplicity, that in the cases where the formation of precipitation in this manner is really possible, super-saturation must first occur, and then precipitation follows.

This assumption was implied by Hann in his above-mentioned memoir* at a time when we still knew nothing as to whether aqueous vapor could actually exist in a supersaturated condition.

* *Zeitschrift Oest. Gesell. Met.*, 1874, vol. IX. [Smithson. Rep., 1877, p. 397.]

But since the possibility of this has been demonstrated by the investigations of Aitken, Coulier, Mascart, Kiessling, and especially by Robert von Helmholtz,* it has some interest for us to make the precipitation from supersaturated air the object of a special investigation.

This precipitation, as is well known, occurs when super-saturated air (which can only exist when perfectly free from dust) is suddenly mixed with very fine particles of solid bodies, or possibly, also, when electric discharges take place through such supersaturated air.† We obtain directly from the above-given rules the amount of the precipitation, as also the rise in temperature.

We have only to omit the parts designated by the indices 1 and 2 in Fig 41, and to consider the condition indicated by the subscript index 3 as the starting point, then the ordinate $T_3 F_3 = y_3$ gives the quantity of water in the state of supersaturation, while $y$ again indicates as before the final remaining moisture; $y_3 - y$ indicates the quantity of moisture precipitated and $t - t_3$ the consequent rise of temperature.

This is, therefore, a method of formation of precipitation, in which one can actually speak of a liberation of latent heat (the latent heat of evaporation), as was formerly done in explaining the formation of precipitation in general.

In a certain sense this usage is allowable, even in the formation of precipitation by mixture, in so far as the temperature of the mixture comes out higher when water is precipitated than when this, under otherwise similar conditions, is not the case because of the insufficient quantity of water. This rise of temperature is however always a very unimportant one in consideration of the small quantity that can be condensed by mixture.

It is otherwise when true super-saturation is present. In such cases the rise of temperature can, according to the degree of super-saturation, be very considerable, as is easily seen from Fig. 41.

Still more considerable must the precipitation be that is caused by the sudden cessation of the super-saturation, namely: So soon as a sudden development of heat occurs at any one place in the atmosphere there follows a powerful ascent of the air which then, by adiabatic cooling, must always produce new formation of precipitation.

Under those conditions, when the vertical distribution of temperature approximates even distantly to that of convective equilibrium, then the sudden cessation of the condition of super-saturation causes this equilibrium to become unstable, and thus this cessation then affords the key to the explanation of a series of phenomena.

I consider it probable that it is in such processes, which indeed deserve a thorough investigation, that we have to seek the reason for the "cloud-bursts" properly so called. Of course, to establish this

* Wiedemann's *Annalen*, 1886, XXVII, p. 527.

† R. von Helmholtz, Wiedemann's *Annalen*, 1887, XXXII, p. 4.

view the proof must first be given that the super-saturation, which we have hitherto only known in laboratory experiments, also occurs in the free atmosphere.

The mixture of super-saturated air with other quantities of air scarcely needs a special consideration, since we at once see the result of such mixture when we imagine, in Fig. 41, one of the points, $F_1$ or $F_2$, transposed to the upper side of the curve $F' F'$, and then execute the further constructions according to the rules previously given.

### (c.) MOIST AIR WITH INTERMIXED WATER OR ICE.

Water occurs in the atmosphere not only as vapor, but also in the form of drops of rain, crystals of ice, and particles of fog. The psychrometer and hygrometer teach us that the air is not necessarily saturated with vapor when water is mixed with it in this manner. Unfortunately we possess only very imperfect data as to how great a quantity of water can in this way be mechanically mixed with the atmosphere.* But there can be scarcely any doubt that the sum of the water mechanically mixed and that which is present in the form of vapor may together be smaller, or equal to, or even greater than the quantity corresponding to saturation for the given temperature. Corresponding to this statement, I will designate such mixtures as air which is "partly saturated mechanically," "wholly saturated mechanically," or "super-saturated mechanically." And now, first of all, we will investigate how such masses of air behave when mixed with ordinary air more or less moist.

By this investigation we shall come to learn the conditions under which the dissolution of fog or clouds or the evaporation of falling raindrops may occur. Such dissolution is, as we at once see, to be attained by mixture only when the intermixed air, which at first may be assumed to be the warmer component, is relatively dry.

Therefore, we will at first investigate the phenomena of mixture under the following conditions:

Let $R_1 > 100$ and composed of two parts, of which the one $\bar{R}_1$ is in the form of vapor and the other $\bar{\bar{R}}_1$ is liquid, and moreover let $\bar{R}_1 < 100$ while

$$\bar{R}_1 + \bar{\bar{R}}_1 = R_1.$$

Furthermore let $R_2 < 100$ and $t_2 > t_1$. This being assumed, the following formulæ hold good, using a notation which by analogy is intelligible of itself:

$$\bar{y}_1 + \bar{\bar{y}}_1 = y_1$$

$$y_1 > y_1'$$

$$\bar{y}_1 < y_1'.$$

*Hann, *Met. Zeit.* 1889, vol. VI, pp. 303–306.

In the accompanying Fig. 43, $\bar{y}_1$ is represented by $T_1\ \bar{F}_1$, and $\bar{\bar{y}}_1$ is represented by $\bar{F}_1\ F_1$, but the remaining lettering certainly needs no further explanation.

However, one thing may be especially noted, that the lines which represent the liquid or frozen * water are limited by two arrow points directed away from each other, since this facilitates a quick comprehension.

If now $m_1$ and $m_2$ are the quantities of the two components that enter into the mixture and we assume here also again that at first both the vapor and also the water are uniformly distributed in the mixture, and that evaporation of the water first occurs afterwards insofar as the saturation of the mixture with vapor allows of any such evaporation, then, just as before, we attain to a state of transition for which the corresponding quantities are appropriately indicated by the subscript 3.

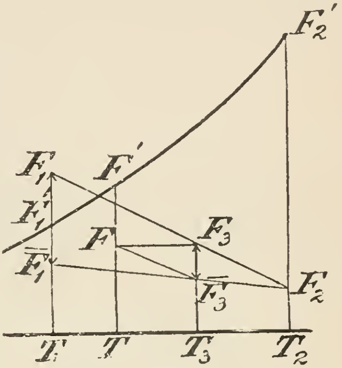


FIG. 43.

The difference between this transition state and the air that is saturated by mixture consists in this, that in the present case the air actually passes through the transition state, whereas in the preceding article it was only imagined for convenience of computation.

In this transition state the quantities $\bar{y}_3$ of vapor and $\bar{\bar{y}}_3$ of water exist in one kilogram of the mixture before the dissolution occurs as is given by the equations

$$\frac{\bar{y}_3-\bar{y}_1}{y_2-\bar{y}_3}=\frac{t_3-t_1}{t_2-t_3}=\frac{m_2}{m_1}$$

and

$$\frac{\bar{\bar{y}}_1-\bar{\bar{y}}_3}{\bar{\bar{y}}_3}=\frac{t_3-t_1}{t_2-t_3}=\frac{m_2}{m_1}$$

which equations lose their apparent want of symmetry when we remember that $\bar{y}_2=y_2$ and that $\bar{\bar{y}}_2=0$.

Moreover, just as before, we have

$$\frac{y_1-y_3}{y_3-y_2}=\frac{t_3-t_1}{t_2-t_3}=\frac{m_2}{m_1}$$

In Fig. 43, $\bar{y}_3$ is represented by the line $T_3\ \bar{F}_3$ and $\bar{\bar{y}}_3$ by the line $\bar{F}_3$ $F_3$.

* In general I assume in what follows that the temperature is above zero, since it is not difficult to modify the considerations appropriately for lower temperatures. But if we would also consider those cases in which water and ice exist alongside of each other or where water is present at temperatures below zero, then the investigation would become inordinately complicated.

The quantity $\bar{y}_3$ that still remains liquid will now dissolve, in so far as the mixture is not saturated, or so much of it will dissolve as is needed for saturation. This of course can only occur in that the mixture itself cools, and that too by 2°.6 or 2°.9 for each gram that is evaporated, since we exclude the cases where heat is communicated from without.

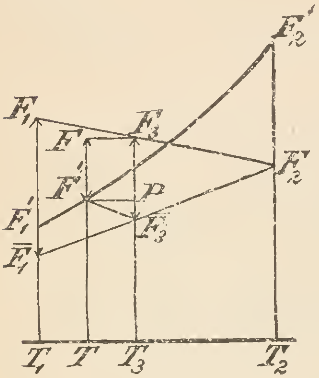



FIG. 44.

The final temperature $T$ is therefore found by passing upward in Fig. 43 from $\bar{F}_3$ toward the left, parallel to the guide-line $F_1$ $F_2$, until at $F$ we reach the same height as $F_3$, or at least until we reach the curve $F'_1$ $F'_2$, in case the line $\bar{F}_3$ $F'$ so drawn would need to cross over the curve $\bar{F}'_1$ $F'_2$.

In this latter case, which is represented in Fig. 44, all the water is not dissolved but only a portion $(y' - \bar{y}_3)$ as is represented in Fig. 44 by the distance $\bar{F}_3$ $P$.

The first of these two cases can be easily handled numerically, since under these conditions we have

$$t = t_3 - K\bar{y}_3$$

$$= t_3 - K\bar{y}_1 \frac{m_2}{m_1 + m_2}$$

$$= \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2} - K\bar{y}_1 \frac{m_2}{m_1 + m_2}$$

$$= \frac{m_1 t_1 + m_2 t_2 - K\bar{y}_1 m_2}{m_1 + m_2}.$$

But the computation is as simple as this only when all the water is really evaporated; in the second case where mechanical supersaturation still continues it is better to apply the graphic method. An especial interest pertains here again to the investigation of the limiting cases for which in general there can occur a complete dissolution of the water originally present as liquid in one of the components. Of such extreme cases there is an extraordinary variety according as we are at liberty to assume arbitrarily either the mixing-ratio or the humidity of one or the other of the components.

At present we shall consider only the question, what is the initial limit of the mixing-ratio for given components in order that complete dissolution must always follow. This limit is evidently obtained when $F'$ $F_3$ lie at the same altitude above the axis of abscissas, that is to say, when $y = y' = y_3$, or when $F$ and $F'$ coincide. In this case $F$ is the

apex of a right-angled triangle whose vertical side is $F_3 \bar{F}_3$ and whose hypothenuse is parallel to the guide-line.

If now we imagine the point $T_3$ moving to and fro along the axis of abscissas, then the apex of the triangle erected in the given manner upon the vertical side $F_3 \bar{F}_3$ will describe a straight line passing through the point $F_2$, which line we easily find when we erect such a triangle on the portion cut off by the straight lines $F_1 F_2$ and $\bar{F}_1 F_2$ from any arbitrary ordinate and then join this apex with $F_2$.

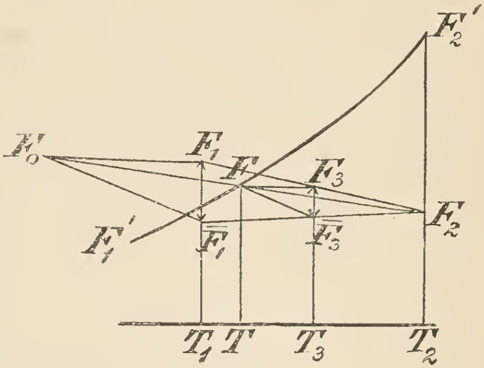


FIG. 45.

We can, for instance, as in Fig. 45, choose for this purpose the ordinate erected at $T_1$.

Then $\bar{F}_1 F_0 F_1$ is the triangle described and $F_0 F_2$ is the straight line on which the desired point $F$ must lie; but since it must also lie on the saturation curve, therefore it is at the intersection of $F_0 F_2$ and the curve $F_1' F_2'$, and the desired limiting value of the mixing ratio is

$$\frac{T_2 T_3}{T_1 T_3} = \frac{m_1}{m_2} = \frac{1}{\mu}.$$

When the mixing ratio attains this limit or exceeds it on the side toward $m_2$, that is to say, as soon as $\frac{m_2}{m_1} =$ or $> \mu$ a complete dissolution of all the suspended water occurs.

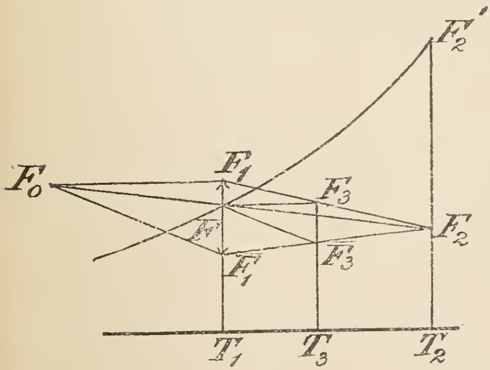


FIG. 46.

In such mixtures it can happen that the line $F F_3$ cuts the curve $F_1' F_2'$ on the left-hand side of $T_1 F_1$. In such cases the temperature resulting from the completion of the mixture is lower than that of either component. The mixing ratio for which this phenomenon begins to occur is easily found by drawing, as in Fig. 46, through $F$ (which is in this case identical with $F_1'$) a line parallel to $F_0 F_1$ (which is a guide line), and find its intersection with $F_1 \bar{F}_2$. The abscissa of this point is then the temperature $t_3$, which is produced by

mixture in this ratio before the subsequent dissolution. From this value of $t_3$ this mixing ratio itself can be determined.

We find by a very simple consideration that for this special value of $t_3$ the following equation holds good:

$$\frac{F_1\bar{F}_1}{F_3\bar{F}_3}=\frac{t_2-t_1}{t_2-t_3}$$

But since

$$F_3\bar{F}_3=\frac{t_3-t_1}{K}$$

and

$$F_1\bar{F}_1=y_1-\bar{y}_1$$

therefore

$$\frac{y_1-\bar{y}_1}{t_3-t_1}K=\frac{t_2-t_1}{t_2-t_3}$$

and consequently

$$\frac{y_1-\bar{y}_1}{t_2-t_1}K=\frac{t_3-t_1}{t_2-t_3}=\frac{m_2}{m_1}\mu_s.$$

Whenever $\mu<\mu_s$, that is to say, when the cooler "mechanically supersaturated air," or at least the saturated component enters into the mixture with greater weight, then will $t<t_1$ and of course $t<t_2$, or in other words, the finally resulting temperature will be lower than that of either component.

These considerations lead to the following apparently very paradoxical result: "*If warmer air is mixed with mechanically saturated or mechanically super-saturated air then a part of the suspended water can be evaporated and thereby a lower temperature produced.*"

"*If the given mechanically saturated air is hygroscopically unsaturated, that is to say, if the vapor is unsaturated, then this lowering of temperature will occur even by the mixing of saturated warmer air (of course in the proper ratio); if the air is saturated as to its vapor and the corresponding mechanical mixture is present as pure super-saturation, then the warmer air must possess a certain degree of dryness that is not difficult to determine.*"

The latter of these two propositions is evident as soon as we allow $\bar{F}_1$ to coincide with $F$ as in Fig. 46, and then with $F_2$ still on the ordinate $T_2 F'_2$ push so far downwards that $F_0F_2$ shall come to lie below $FF_2$, a condition which, however, is only to be satisfied so long as $T_2$ does not have too high a value.

The very paradoxical sentences just set forth lose their extraordinary appearance as soon as we recognize that a mixture of water and unsaturated moist air is not in a condition of equilibrium, but that in such a mixture evaporation must always occur unless by some special process the condition is kept stationary. Such mixtures are exemplified in the fogs and clouds, and during rain. The behavior of such mixtures has been implicitly investigated in the previous paragraphs and now only a few words further need be said.

Perhaps one might consider it a theoretical error that this behavior was not from the very first made the object of investigation but that the mixture of such masses with other air was chosen as the starting point of this study. But on the one hand this was the way by which I was actually led to the whole subject, and on the other hand abbreviations and simplifications are hereby rendered possible that seem to me sufficiently important to justify retaining this arrangement of the subject-matter.

In order to study the behavior of such mixtures when left to themselves we have only to choose as a starting point the condition represented by the ordinate $T_3\ F_3$ in Fig. 44, there considered as a state of transition, and we arrive then, according to the same rules as above, to the final condition $T\ F$ and thus also to the final temperature $T$.

Hence we recognize immediately "*that mixtures of water and unsaturated air as soon as left to themselves must cool and so much the more the further the vapor is from the point of saturation and the more liquid water or ice is mixed with it.*"

These considerations explain a phenomenon that I had frequently observed, but concerning which I was until recently not certain whether it might not be of a purely subjective nature.

It had frequently happened to me on passing through strata of fog such as fill the mountain valleys in the mornings of calm and subsequently clear days, that the impression of more severe cold occurs precisely when we attain the upper limit of the fog as we ascend the valley. Again it frequently occurred that just before the sun dissipates the morning fog that collects in valleys or spreads over the lowlands, the sensation of especial cold is experienced.

Such impressions of the sensations can of course very easily lead to error; but according to what has been above said, it is probable from theoretical grounds also, that the temperature just below the upper boundary of a dissolving layer of fog is lower than that of the layers above and below it. For when the sun begins to do its work at the upper boundary of the fog there then occurs, first, relative dryness immediately above this, and this relative dryness will, according to the velocity with which evaporation of the fog particles takes place, partly by diffusion, partly by direct radiation, also propagate itself to a certain, although very moderate depth, in the layer of fog.

But thereby (at least in many cases) the evaporation will be more accelerated than would correspond to the increase of heat by direct radiation, that is to say, the temperature must sink.

This expectation, grounded partly on the impression of one's feeling, partly on theoretical physical considerations, has now, during the writing of this memoir, received a confirmation by actual measurements. For the communication of these measurements I have to thank Mr. Bartsch von Sigsfeld, who, in a balloon constructed at his own expense, has already undertaken many aerial voyages for scientific purposes

and has also carried out meteorological observations when ascending mountains.

I will here first quote the results that von Sigsfeld obtained during a balloon trip from Augsburg on October 26, 1889, on which occasion the remarkably perfected aspiration psychrometer of Assmann of the newest construction * was used for the determination of temperatures and moisture.

The start occured as above remarked on October 26, at about 15 minutes before 10 A. M. The landing took place about 3 P. M., near Plochingen, on the railroad line between Ulm and Stuttgart. The general character of the weather on this day can be summarized as follows: While an extended barometric maximum with a center closely clinging to the Scandinavian peninsula, covered all of northern and central Europe there prevailed over southwestern and southern Europe a region of depression that starting from the southwest extended over all lands bordering on the Mediterranean and sent individual outrunners into southern Germany. The weather was almost everywhere cloudy with moderate motion of the air from the east.

Above Southern Germany itself there floated a layer of dry upper haze, whose lower indefinite boundary lay at an altitude of about 600 metres above the sea, whereas the upper, very well defined and flat boundary was found at an altitude of 1,200 metres. Up to this latter altitude the wind blew with moderate strength from the northeast, but above this it blew strongly from the south-southeast. Unfortunately only a few observations could be made, since von Sigsfeld was to a very large extent occupied with the navigation of the balloon, but notwithstanding this some important data were secured which I here reproduce:

| Local time. | Altitude above sea level. | Barometric pressure. | Temperature. | | Relative humidity. | Remarks. |
|---|---|---|---|---|---|---|
| | | | Dry bulb. | Wet bulb. | | |
| *h. m.* | *Metres.* | *mm.* | ° *C.* | ° *C.* | *Per cent.* | |
| 9.47 | 471 | 723.3 | 7.5 | 6.2 | 83 | At the starting place.† |
| 10.45 | 471 | .......... | .......... | .......... | .......... | Start: The temperature had changed very little since preceding observation. |
| (?) | (?) | .......... | 3.0 | 2.9 | 98 | Close under the upper boundary of the fog [or high dry haze]. |
| (?) | 1202 | 660.0 | .......... | .......... | .......... | Upper boundary of the haze. |
| 11.15 | 1615 | 630.0 | 5.3 | 2.0 | 55 | |
| 12.20 | 1358 | 617.3 | 5.5 | 3.5 | 73 | After this a further rise of the balloon up to about 2,900 metres altitude and a steady diminution of the temperature of the dry bulb to about 3° C. |

* Assmann. Das Aspirationspsychrometer. *Zeitschrift f. Luftschiffahrt*, 1890, IX, pp. 1–9 and 30–38.
† The barometer of the Augsburg Meteorological Station is at the altitude 499.6 metres.

These numbers, few as they are, still show that the layer of fog just under its upper boundary was the coolest portion of the whole path traversed by the balloon, and that close above this boundary the temperature shows a rapid rise, but the relative humidity a rapid fall.

Von Sigsfeld had obtained similar results even earlier, namely, October 5, 1887, on the occasion of an ascent of the Faulhorn, in Aargau, for the purpose of taking photographs.

I give the appropriate data in the following table:

| Time. | Altitude. | Barometer. | Dry bulb. | Wet bulb. | Relative humidity. | Remarks. |
|---|---|---|---|---|---|---|
| *h. m.* | *Metres.* | *mm.* | ° | ° | *Per cent.* | |
| 8 0 a. m. | 808 | 692. 0 | 4. 6 | ......... | ......... | Oberstdorf.* |
| 8 0 a. m | 1, 058 | 671. 5 | 2. 1 | 1. 9 | 96 | Starting point, Riezlern; fog. |
| 8 30 a. m. | 1, 203 | 659. 1 | 0. 7 | 0. 7 | 100 | Upper limit of fog. |
| 9 10 a. m | 1, 487 | 636. 7 | 6. 2 | 3. 5 | 64 | |
| 10 05 a. m. | .......... | 609. 1 | 12. 0 | 8. 0 | .......... | |
| 10 40 a. m. | 2, 029 | 596. 0 | 8. 5 | 4. 0 | 48 | Faulhorn. |
| 2 15 p. m. | 2, 031 | 595 1 | 4. 6 | 4. 4 | 97 | Faulhorn; fog rises; upper boundary of fog attains and surpasses the summit of the mountain. |
| 2 30 p. m. | 2, 029 | 594. 8 | 3. 6 | 2. 8 | 89 | Faulhorn; fog sinks. |
| 2 35 p. m | .......... | 594. 8 | 5. 2 | 4. 2 | 86 | |
| 2 40 p. m | .......... | 594. 8 | 3. 2 | 3. 2 | 100 | |
| 3 12 p. m | .......... | 594. 8 | 2. 5 | 2. 5 | 100 | Fog sinks; upper boundary descends to the summit of the mountain. |
| 3 50 p. m | 1, 950 | 600. 0 | 2. 0 | 2. 0 | 100 | Descending; fog still continues. |
| 4 10 p. m. | 1, 785 | 612. 3 | 2. 0 | 2. 0 | 100 | Fog. |
| 4 20 p. m. | 1, 668 | 621. 0 | 2. 8 | 2. 8 | 100 | Fog. |
| 4 25 p. m. | 1, 615 | 625. 0 | 3. 5 | 3. 4 | 98 | Above the lower limit of fog. |
| 4 25 p. m. | .......... | 625. 0 | 4. 5 | 4. 0 | 93 | Below the lower limit of fog. |
| 5 10 p. m. | 1. 078? | 668. ? | 6. 7 | 6. 2 | 93 | At Riezlern.† |

* According to Trautwein ("Southern Bavaria, etc.," seventh edition, Augsburg, 1884) the altitude above the sea of Oberstdorf, which is that here adopted, is 808 metres; that of the summit of the Faulhorn is 2,033 metres, so that the value 2,031 metres is an excellent testimony to the reliability of the data.

† The morning observation gave the altitude of Riezlern as 1,058 metres, whereas the observation at 5 hours 10 minutes p. m. gave 1,078 metres, making use of the barometric pressure observed in the morning in Oberstdorf. But if we assume, as is required by the observations at Augsburg and Munich, that this reading had, during the intervening time, diminished by 1 millimetre, and furthermore adopt the very probable assumption that the aneroid could not perfectly follow the rapid changes of pressure during the descent, and therefore read about 2 millimetres too low, we shall obtain for Riezlern the altitude 1,062 metres above the sea, or a figure that agrees almost perfectly with that deduced from the morning observation.

The numbers above tabulated were given on the one hand with a well compared, quite reliable aneroid, and on the other so far as concerns the temperatures with an Assmann's aspiration psychrometer of the older construction.

From the above table we see very clearly that the upper boundary of the stratum of fog always shows a lower temperature than the neighboring strata above and below.

But whether this is as above assumed essentially the cold due to evaporation can not properly be decided. The high relative humidity

which was found even in the highest layer of fog raises some doubt in this direction. Some observations made by First Lieutenant Moedebeck and Lieutenant Gross on the occasion of a balloon voyage made on June 19, 1889, and which Lieutenant Gross has recently published * in a very interesting essay, apparently speak more clearly on this point, and certainly deserve a thorough scientific analysis. †

Here also the passage through thick clouds showed that the temperature at the upper boundary of these fell very low but immediately above this it rose at once suddenly. The observations of humidity also agree better with the theoretical views developed above. On this point Lieutenant Gross says with reference to a diagram given by him which shows the changes of the dry and wet thermometers, "We see from the comparison of the curves of the dry and wet thermometers that the moisture of the air rapidly increases with approach to the cloud, and that in the cloud itself where both curves coincide the air is completely saturated with aqueous vapor. But it is only in the lower part of the cloud that this is the case, and the moisture diminishes towards its upper part, an observation that I have already frequently made. This is certainly also explicable: In the upper part of the cloud the sun acts again as at first. Immediately above the cloud the wet thermometer makes a sudden rise. The air becomes suddenly very dry, as results without anything further, from the heat reflected back from the cloud."

That the lowest temperature should be observed immediately under the upper boundary of the cloud in spite of the influence of the sun seems to me explicable only by means of the cold due to evaporation in accordance with the manner above theoretically predicted.

One ought to be able to observe with all sharpness on the Eiffel tower the questions relating to the behavior of the upper surface of fog since it must frequently happen there that the boundary floats but a short distance above the meteorological instruments.

Perhaps also it will be possible there to establish at different heights self-registering thermometers and psychrometers or hygrometers in order to obtain truly simultaneous observations immediately above and below the upper boundary of the fog (or mist).

### (*d.*) THE FORMATION AND DISSOLUTION OF FOG AND CLOUD.

The preceding investigations into the formation of precipitation by mixture of quantities of air of unequal warmth and moisture show that

* *Zeitschrift für Luftschiff fahrt*, 1889, VIII, p. 249.

† In referring to this essay I might also mention that Lieutenant Gross has also in the meantime confirmed the expectation expressed in my former communication [see pages 251–253] according to which the inversion of temperature in the region of the winter anti-cyclone is not a peculiarity of mountainous regions. On the occasion of a balloon voyage undertaken on December 19, 1888, from Berlin under the influence of such an anti-cyclone, the sling thermometer gave an increase of temperature of 8° in 1,000 metres of ascent between 1 P. M. and 4 P. M.

although such mixtures can not produce heavy rain or snow yet they can be of great importance in the formation of fog and cloud.

In accordance with this there are three processes that can, either by themselves alone or in conjunction, cause a condensation of the aqueous vapor in the atmosphere:

(*a*) Direct cooling, whether by contact with cold bodies or by radiation.

(*b*) Adiabatic expansion, or at least expansion with insufficient addition of heat.

(*c*) Mixture of masses of air of different temperatures.

In a corresponding manner the dissolution of fog and cloud already present may take place through the following processes:

(*a*) Direct warming, either by radiation or by contact with warmer bodies.

(*b*) Compression, whether adiabatic or at least with an insufficient abstraction of heat.

(*c*) Mixture with other masses of air having sufficient temperature and moisture.

Of these three different processes the one first mentioned is always the most effective.

In order to condense or dissolve a given quantity of water there need be only a relatively slight direct cooling or warming. When the condensation or dissolution of a certain quantity is to be accomplished by adiabatic expansion or compression the cooling or warming must be greater, that is to say, must cover a wider range of temperature than for direct cooling or warming.

Still larger temperature differences must come into play when the same quantity is to be condensed or evaporated by the process of mixture, in so far as this is any way possible.

The first pair of these processes, namely, the direct cooling or direct warming, comes especially into consideration in the formation of fog proper, which beginning at the earth's surface, extends upwards to greater or less altitudes. At times of excessive radiation the earth's surface first cools. When the cooling has reached the dew point there occurs condensation in the very lowest layer. Hereby the emissivity of this layer itself is increased. It then cools in its upper portion also by radiation, and thus the layer of fog grows upwards more and more until subsequently, at the time of increased inflow of heat, it dissolves itself in a precisely inverse manner.

No other considerable precipitation is formed by this method of condensation except the so-called drizzle. The reason of this undoubtedly is that the growth of the layer of fog upwards removes the possibility of further more intense radiation by the lower stratum. In the higher strata of the atmosphere such condensation by direct radiation can certainly only occur when cloudiness has already been produced in some other way, whether by mixture or by expansion or possibly by smoke.

At the upper limit of the cloud, especially in stratus clouds, the processes of growth and dissolution of the cloud by direct loss or gain of heat by radiation are carried on like the formation and dissolution of fog in the lowest strata of air.

The formation of clouds by adiabatic expansion as well as the dissolution by compression occurs wherever we have to do with ascending or descending currents of air. This process has in recent times been so frequently treated that the subject may here be treated very briefly. The cumulus clouds of summer with horizontal bases, the thunder cloud and the rain cloud, properly so called, owe their origin to this process. To what extent "nocturnal radiation" influences the upper layers of such clouds can only be made clear by further investigation.

Still more complicated than the two methods hitherto considered in the formation and dissolution of clouds and fog are the processes that accompany mixture. In both the above mentioned pairs of processes a steady increase of cooling or warming is accompanied by a steadily progressive condensation or dissolution. It is quite otherwise in mixtures. A process of mixture can progress in the same direction and yet cause at first condensation and in its subsequent stages dissolution. The breath which we exhale into the cool air leaves the mouth saturated but not yet in the condition of fog; only after the beginning of the mixing with the colder air does the formation of the cloud of vapor begin, which then through further mixture with colder, drier air, again dissolves. We see this process depicted in a strictly mathematical way in Fig. 38. If for instance we assume that a small quantity of air at the temperature $t_1$ is mixed with a larger quantity at the higher temperature $t_2$, then all possible mixing-ratios will occur from $\frac{m_2}{m_1}=0$ up to the final result, which we will assume to be greater than that which corresponds to the higher value $y_2^*$. In this case the quantity of contained water $y$ passes through all values belonging to the ordinates of the line $F_1F_2$ until reaching the final value $y>y_2^*$. In this process condensation must occur as soon as the mixing-ratio exceeds the value which corresponds to the ordinate $y_1^*$; if it increases still further then beyond a definite point as it approaches towards the ordinate $y_2^*$ dissolution again begins, which becomes complete for a mixing-ratio corresponding to the ordinate $y_2^*$ and thus again results an unsaturated mixture.

If a smaller quantity of nearly saturated warmer air mixes with a larger quantity of colder air then will the mixture pass through its conditions in an inverse order, and again the initial condensation and the subsequent dissolution will occur under the conditions assumed in Fig. 38.

Although now in both cases condensation occurs first and then dissolution, still there is an important difference between them. For if we imagine the mixing-ratio to undergo steady change between the points

of condensation and of dissolution, that is to say, between the ordinates $y_1^*$ and $y_2^*$, then will the resulting mean temperature $t=\frac{t_1^*+t_2^*}{2}$ be attained quicker when we go from $y_1^*$ towards $y_2^*$ than when we go from $y_2^*$ towards $y_1^*$. For since $t > t_3$ therefore for $t = \frac{1}{2}(t_1^*+t_2^*)$ the mixing-ratio $\frac{m_1}{m_2} > 1$, that is to say, the mixture shows the average temperature, although so far as mass is concerned the colder component is in excess. According to this, if we mix saturated cooler air with steadily increasing quantities of saturated warmer air, then the warming of the mixture proceeds more rapidly at first than subsequently, whereas in the reverse process cooling proceeds more slowly at first and then steadily faster. The quantity condensed has also a similar relation; it also attains its maximum when there is an excess of the cooler component.

"*Therefore condensation begins sooner when a jet of cold moist air penetrates a large mass of warmer air than when a jet of warm moist air is blown into cooler air.*"

Therefore by the outward appearances of clouds that are forming and dissolving in this manner, one perceives whether warmer or colder air predominates.

From all the preceding we conclude that the following forms of fog and clouds may be considered as originating by mixture:

(1) The fog above warm moist surfaces, under the influence of colder air, therefore especially the fog over the sea in the cold season of the year or during the occurrence of cold winds.

(2) The "rank and file" clouds occurring on the boundary between two different strata of air flowing rapidly above each other, which von Helmholtz* has first recognized as a consequence of wave motion and designated by the name, atmospheric billows, in which however adiabatic condensation also comes into consideration at places where the air is thrown upward after the manner of the formation of crests and foam on ocean waves.

(3) The layers of stratus that also form at such separating surfaces and which frequently first appear as atmospheric billows and subsequently become denser.

(4) Cloud streamers that form and again dissolve at the summits of mountains or in narrow mountain p sses when the form of the mountain is such as to make it possible for jets of warmer or colder masses of air to penetrate into similar masses of other temperatures.†

(5) The ragged clouds, or the disconnected clouds, such as one frequently observes during rapid motions of the air, perpetually changing

---

* *Sitzungsberichte, König. Preus. Akad. Wissensch. zu Berlin*: Berlin, 1888, p. 661, and 1889, p. 503. [See also Nos. VI and VII of this collection.]

† Von Bezold, *Himmel und Erde*, 1889, vol. II, p. 7.

their form and appearing and disappearing, and such as also occur with clouds formed by adiabatic expansion, especially during thunder storms.

These different methods of cloud formation by direct cooling, by adiabatic expansion, and by mixture can of course also occur side by side in the most varied combinations, as is expressed in the extraordinary diversity of cloud forms.

It seems to me very important in the study of these forms to keep these different processes in view, since only then can we hope finally to attain a thorough knowledge of these forms.

Above all, as Hellmann has appropriately expressed it, it is necessary to lay the foundation for a "physiology of the clouds" before we can hope to attain to a truly satisfactory arrangement and nomenclature.*

But further work will still be necessary before this problem is solved, since on the one hand the question becomes more complicated the nearer we approach to it, and since on the other hand it appears so extraordinarily difficult to realize experimentally even approximately those conditions under which the formation and dissolution of clouds take place in the atmosphere.

Beautiful and praiseworthy as are the experiments that Vettin has made with clouds of smoke, still we must be very careful about the conclusions which we would draw from them as to the formation of the real clouds. All experiments with smoke, when looked at properly, give only pictures of the movements in dry air, since the condensation and evaporation as well as the processes of compression and expansion are excluded, and we therefore are working under conditions such that in the real atmosphere no formation of clouds would occur.

But it is precisely because of these processes (condensation, evaporation, compression, and expansion) that we can not consider the motion of a cloud as a measure of the motion of the air, for not only do clouds hang apparently motionless on the mountains, whereas in fact strong winds are streaming through them (*e. g.* Föhn cloud-bank, the Table-cloth of the Table mountain, the Cloud-cap of the Helm-wind) but it even happens to aëronauts that they pass through clouds while moving in a horizontal direction. This latter is however only possible when the cloud has a velocity different from that of the air in which it floats. since the balloon itself has only the power of vertical motion.

The cloud is in fact not a body that can be driven forward as such by the air unchanged, but is a form in a process of continuous formation and disappearance, and can have as a whole motions entirely different from those of the particles of which it consists.

On account of the increased interest with which at the present time we are studying the forms and motions of the clouds, it seemed to me important to call attention to all these points since we must have these in mind when we attempt from the external appearance of the clouds to draw any conclusion as to the processes which in individual cases determine their growth or dissolution and therefore also their form.

* Compare also O. Volger in *Gaea*, 1890, vol. II, pp. 65–75.

## APPENDIX.

*Table giving the quantity of water in grams that is contained as vapor in a kilogram of saturated air.*

| t | b=760mm | b=700mm | b=600mm | b=500mm | b=400mm | b=300mm | b=200mm |
|---|---|---|---|---|---|---|---|
| −30 | 0.31 | 0.34 | 0.39 | 0.48 | 0.60 | 0.80 | 1.20 |
| 29 | .34 | .37 | .43 | .52 | .65 | .87 | 1.31 |
| 28 | .38 | .41 | .48 | 57 | .71 | .95 | 1.43 |
| −27 | 0.41 | 0.45 | 0.52 | 0.63 | 0.78 | 1.04 | 1.56 |
| 26 | .45 | .49 | .57 | .69 | .86 | 1.14 | 1.71 |
| 25 | .49 | .54 | .63 | .75 | .94 | 1.25 | 1.88 |
| 24 | .54 | .59 | .69 | .82 | 1.03 | 1.37 | 2.06 |
| 23 | .59 | .65 | .75 | .90 | 1.13 | 1.50 | 2.25 |
| −22 | 0.65 | 0.71 | 0.82 | 0.99 | 1.23 | 1.63 | 2.46 |
| 21 | .71 | .77 | .90 | 1.08 | 1.34 | 1.78 | 2.69 |
| 20 | .77 | .84 | .98 | 1.18 | 1.46 | 1.94 | 2.94 |
| 19 | .84 | .92 | 1.07 | 1.28 | 1.60 | 2.12 | 3.21 |
| 18 | .92 | 1.00 | 1.16 | 1.39 | 1.74 | 2.32 | 3.50 |
| −17 | 1.00 | 1.09 | 1.26 | 1.52 | 1.90 | 2.53 | 3.81 |
| 16 | 1.09 | 1.18 | 1.37 | 1.65 | 2.07 | 2.75 | 4.14 |
| 15 | 1.19 | 1.28 | 1.49 | 1.79 | 2.24 | 2.99 | 4.49 |
| 14 | 1.28 | 1.39 | 1.62 | 1.94 | 2.43 | 3.24 | 4.87 |
| 13 | 1.39 | 1.51 | 1.76 | 2.11 | 2.64 | 3.52 | 5.28 |
| −12 | 1.50 | 1.64 | 1.90 | 2.29 | 2.86 | 3.82 | 5.73 |
| 11 | 1.63 | 1.77 | 2.06 | 2.48 | 3.10 | 4.13 | 6.20 |
| 10 | 1.76 | 1.91 | 2.23 | 2.68 | 3.35 | 4.47 | 6.72 |
| 9 | 1.91 | 2.07 | 2.41 | 2.90 | 3.62 | 4.84 | 7.26 |
| 8 | 2.06 | 2.24 | 2.61 | 3.13 | 3.92 | 5.23 | 7.85 |
| −7 | 2.23 | 2.42 | 2.82 | 3.38 | 4.24 | 5.65 | 8.49 |
| 6 | 2.40 | 2.61 | 3.04 | 3.65 | 4.58 | 6.10 | 9.16 |
| 5 | 2.59 | 2.81 | 3.28 | 3.94 | 4.94 | 6.58 | 9.88 |
| 4 | 2.79 | 3.03 | 3.54 | 4.25 | 5.32 | 7.09 | 10.66 |
| 3 | 3.01 | 3.27 | 3.81 | 4.58 | 5.72 | 7.64 | 11.49 |
| −2 | 3.24 | 3.52 | 4.10 | 4.93 | 6.16 | 8.23 | 12.37 |
| −1 | 3.48 | 3.78 | 4.42 | 5.30 | 6.63 | 8.85 | 13.32 |
| 0 | 3.75 | 4.07 | 4.75 | 5.71 | 7.13 | 9.52 | 14.33 |
| +1 | 4.03 | 4.37 | 5.10 | 6.13 | 7.67 | 10.24 | .......... |
| 2 | 4.32 | 4.70 | 5.48 | 6.58 | 8.24 | 11.00 | .......... |
| +3 | 4.64 | 5.04 | 5.88 | 7.07 | 8.85 | 11.81 | .......... |
| 4 | 4.98 | 5.41 | 6.31 | 7.58 | 9.49 | 12.68 | .......... |
| 5 | 5.34 | 5.80 | 6.77 | 8.13 | 10.18 | 13.60 | .......... |
| 6 | 5.71 | 6.22 | 7.26 | 8.72 | 10.91 | .......... | .......... |
| 7 | 6.13 | 6.66 | 7.77 | 9.34 | 11.69 | .......... | .......... |
| +8 | 6.56 | 7.13 | 8.32 | 9.99 | 12.52 | .......... | .......... |
| 9 | 7.02 | 7.63 | 8.91 | 10.70 | 13.40 | .......... | .......... |
| 10 | 7.51 | 8.16 | 9.53 | 11.44 | 14.33 | .......... | .......... |
| 11 | 8.03 | 8.72 | 10.18 | 12.24 | 15.32 | .......... | .......... |
| 12 | 8.58 | 9.32 | 10.88 | 13.08 | 16.38 | .......... | .......... |
| +13 | 9.16 | 9.95 | 11.62 | 13.97 | 17.50 | .......... | .......... |
| 14 | 9.78 | 10.62 | 12.41 | 14.91 | 18.69 | .......... | .......... |
| 15 | 10.43 | 11.34 | 13.24 | 15.91 | 19.94 | .......... | .......... |
| 16 | 11.13 | 12.09 | 14.12 | 16.97 | .......... | .......... | .......... |
| 17 | 11.86 | 12.89 | 15.05 | 18.10 | .......... | .......... | .......... |

*Table giving the quantity of water in grams that is contained as vapor in a kilogram of saturated air*—Continued.

| $t$ | $b=760^{mm}$ | $b=700^{mm}$ | $b=600^{mm}$ | $b=500^{mm}$ | $b=400^{mm}$ | $b=300^{mm}$ | $b=200^{mm}$ |
|---|---|---|---|---|---|---|---|
| +18 | 12.64 | 13.73 | 16.04 | 19.29 | ........ | ........ | ........ |
| 19 | 13.46 | 14.62 | 17.09 | 20.55 | ........ | ........ | ........ |
| 20 | 14.33 | 15.57 | 18.20 | 21.88 | ........ | ........ | ........ |
| 21 | 15.25 | 16.57 | 19.37 | ........ | ........ | ........ | ........ |
| 22 | 16.22 | 17.63 | 20.59 | ........ | ........ | ........ | ........ |
| +23 | 17.24 | 18.75 | 21.90 | ........ | ........ | ........ | ........ |
| 24 | 18.32 | 19.93 | 23.28 | ........ | ........ | ........ | ........ |
| 25 | 19.47 | 21.17 | 24.73 | ........ | ........ | ........ | ........ |
| 26 | 20.68 | 22.48 | ........ | ........ | ........ | ........ | ........ |
| 27 | 21.95 | 23.86 | ........ | ........ | ........ | ........ | ........ |
| +28 | 23.29 | 25.31 | ........ | ........ | ........ | ........ | ........ |
| 29 | 24.70 | 26.84 | ........ | ........ | ........ | ........ | ........ |
| 30 | 26.18 | 28.47 | ........ | ........ | ........ | ........ | ........ |

In computing this table the vapor tensions of aqueous vapor have been adopted as given by Broch, Travaux et Mémoirés, Bur. Internat. des Poids et Mesures, 1881, tome I.

XVIII.

# ON VIBRATIONS OF AN ATMOSPHERE.*

By Lord Rayleigh.

In order to introduce greater precision into our ideas respecting the behavior of the earth's atmosphere, it seems advisable to solve any problems that may present themselves, even though the search for simplicity may lead us to stray rather far from the actual question. It is proposed here to consider the case of an atmosphere composed of gas which obeys Boyle's law, viz, such that the pressure is always proportional to the density. And in the first instance we shall neglect the curvature and rotation of the earth, supposing that the strata of equal density are parallel planes perpendicular to the direction in which gravity acts.

If $p$, $\sigma$ be the equilibrium pressure and density at the height $z$, then

$$\frac{dp}{dz}=-\sigma g; \qquad (1)$$

and by Boyle's law,

$$p=a^2\sigma, \qquad (2)$$

where $a$ is the velocity of sound. Hence

$$\frac{d\sigma}{\sigma dz}=-\frac{g}{a^2}, \qquad (3)$$

and

$$\sigma=\sigma_0 e^{\frac{-gz}{a^2}}, \qquad (4)$$

where $\sigma_0$ is the density at $z=0$. According to this law, as is well known, there is no limit to the height of the atmosphere.

Before proceeding further, let us pause for a moment to consider how the density at various heights would be affected by a small change of temperature, altering $a$ for $a'$, the whole quantity of air and there-

* From the *London, Edinburgh and Dublin Phil. Mag.*, Feb., 1890, fifth series, vol. XXIX, pp. 173–180.

fore the pressure $p_0$ at the surface remaining unchanged. If the dashes relate to the second state of things, we have

$$\sigma=\sigma_0\, e^{\frac{-gz}{a^2}},\quad \sigma'=\sigma'_0\, e^{\frac{-gz}{a'^2}},$$

$$p=p_0\, e^{\frac{-gz}{a^2}},\quad p'=p_0\, e^{\frac{-gz}{a'^2}},$$

while

$$a^2\sigma_0=a'^2\sigma'_0.$$

If $a'^2-a^2=\delta a^2$, we may write approximately

$$\frac{p'-p}{p_0}=\frac{\delta a^2}{a^2}\frac{gz}{a^2}\, e^{\frac{-gz}{a^2}}.$$

The alteration of pressure vanishes when $z=0$, and also when $z=\infty$. The maximum occurs when $\frac{gz}{a^2}=1$, that is, when $p=\frac{p_0}{e}$. But $(p'-p_0)$ increases relatively to $\sigma$, continually with $z$.

Again, if $\rho$ denote the proportional variation of density,

$$\rho=\frac{\sigma'-\sigma}{\sigma}=\frac{a^2}{a'^2}\left(e^{\frac{-gz}{a'^2}+\frac{gz}{a^2}}-1\right).$$

If $a'^2>a^2$, $\rho$ is negative when $z=0$, and becomes $+\infty$ when $z=\infty$. The transition $\rho=0$ occurs when $\frac{gz}{a^2}=1$, that is, at the same place where $p'-p$ reaches a maximum.

In considering the small vibrations, the component velocities at any point are denoted by $u$, $v$, $w$, the original density $\sigma$ becomes $(\sigma+\sigma\rho)$, and the increment of pressure is $\delta p$. On neglecting the squares of small quantities the equation of continuity is

$$\sigma\frac{d\rho}{dt}+\sigma\frac{du}{dx}+\sigma\frac{dv}{dy}+\sigma\frac{dw}{dz}+w\frac{d\sigma}{dz}=0$$

or by (3)

$$\frac{d\rho}{dt}+\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}-\frac{gw}{a^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The dynamical equations are

$$\frac{d\delta p}{dx}=-\sigma\frac{du}{dt},\quad \frac{d\delta p}{dy}=-\sigma\frac{dv}{dt},\quad \frac{d\delta p}{dz}=-g\sigma\rho-\sigma\frac{dw}{dt};$$

or by (3) since

$$\delta p=a^2\sigma\rho,$$

$$a^2\frac{d\rho}{dx}=-\frac{du}{dt},\quad a^2\frac{d\rho}{dy}=-\frac{dv}{dt},\quad a^2\frac{d\rho}{dz}=-\frac{dw}{dt} \quad . \quad . \quad . \quad . \quad (6)$$

We will consider first the case of one dimension, where $u$, $v$ vanish, while $\rho$, $w$ are functions of $z$ and $t$ only. From (5) and (6),

$$\frac{d\rho}{dt}+\frac{dw}{dz}-\frac{gw}{a^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$a^2\frac{d\rho}{dz}=-\frac{dw}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or by elimination of $\rho$,

$$\frac{1}{a^2}\cdot\frac{d^2w}{dt^2}=\frac{d^2w}{dz^2}-\frac{g}{a^2}\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The right-hand member of (9) may be written

$$\left(\frac{d}{dz}-\frac{g}{2a^2}\right)^2 w-\frac{g}{4a^4}w,$$

and in this the latter term may be neglected when the variation of $w$ with respect to $z$ is not too slow. If $\lambda$ be of the nature of the wave-length, $\frac{dw}{dz}$ is comparable with $\frac{w}{\lambda}$; and the simplification is justifiable when $a^2$ is large in comparison with $g\lambda$, that is when the velocity of sound is great in comparison with that of gravity-waves (as upon water) of wave length $\lambda$. The equation then becomes

$$\frac{d^2w}{dt^2}=a^2\left(\frac{d}{dz}-\frac{g}{2a^2}\right)^2 w;$$

or, if

$$w=We^{\frac{\frac{1}{2}gz}{a^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{d^2W}{dt^2}=a^2\frac{d^2W}{dz^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the ordinary equation of sound in a uniform medium. Waves of the kind contemplated are therefore propagated without change of type except for the effect of the exponential factor in (10), indicating the increase of motion as the waves pass upwards. This increase is necessary in order that the same amount of energy may be conveyed in spite of the growing attenuation of the medium. In fact $w^2\sigma$ must retain its value, as the waves pass on.

If $w$ vary as $e^{int}$, the original equation (9) becomes

$$\frac{d^2w}{d^2z}-\frac{g}{a^2}\frac{dw}{dz}+\frac{n^2w}{a^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let $m_1$, $m_2$ be the roots of

$$m^2-\frac{g}{a^2}m+\frac{n^2}{a^2}=0,$$

so that

$$m = \frac{g \pm \sqrt{(g^2 - 4n^2 a^2)}}{2a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then the solution of (12) is

$$w = Ae^{m_1 z} + Be^{m_2 z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$A$ and $B$ denoting arbitrary constants in which the factor $e^{int}$ may be supposed to be included.

The case already considered corresponds to the neglect of $g^2$ in the radical of (13), so that

$$m = \frac{g \pm 2\, nai}{2\, a^3}$$

and

$$we^{-\frac{\frac{1}{2}gz}{a^2}} = Ae^{in\left(t + \frac{z}{a}\right)} + Be^{in\left(t - \frac{z}{a}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

A wave propagated upwards is thus

$$w = e^{\frac{\frac{1}{2}gz}{a^2}} \cos n\left(t - \frac{z}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and there is nothing of the nature of reflection from the upper atmosphere.

A stationery wave would be of type

$$w = \mathrm{e}^{\frac{\frac{1}{2}gz}{a^2}} \cos nt \sin \frac{nz}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$w$ being supposed to vanish with $z$. According to (17), the energy of vibration is the same in every wave length, not diminishing with elevation. The viscosity of the rarefied air in the upper regions would suffice to put a stop to such a motion, which can not therefore be taken to represent anything that could actually happen.

When $2\, na < g$, the values of $m$ from (13) are real, and are both positive. We will suppose that $m_1$ is greater than $m_2$. If $w$ vanish with $z$, we have from (14) as the expression of the stationary vibration

$$w = \cos nt \left(e^{m_1 z} - e^{m_2 z}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which shows that $w$ is of one sign throughout. Again by (8)

$$a^2 \rho = n \sin nt \left\{ \frac{e^{m_1 z}}{m_1} - \frac{e^{m_2 z}}{m_2} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Hence $\frac{d\rho}{dz}$, proportional to $w$, is of one sign throughout; $\rho$ itself is negative for small values of $z$, and positive for large values, vanishing once when

$$e^{(m_1 - m_2)z} = \frac{m_1}{m_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When $n$ is small we have approximately

$$\left.\begin{aligned} m_1 &= \frac{g}{a^2} - \frac{n^2}{g}, \\ m_2 &= \frac{n^2}{g} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

so that $\rho$ vanishes when

$$e^{\frac{gz}{a^2}} = \frac{g_2}{n^2 a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or by (4) when

$$\frac{\sigma}{\sigma_0} = \frac{n^2 a^2}{g^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Below the point determined by (23) the variation of density is of one sign and above it of the contrary sign. The integrated variation of density, represented by $\int_0^{\infty} \sigma\,\rho\,dz$, vanishes, as of course it should do.

It may be of interest to give a numerical example of (23). Let us suppose that the period is one hour, so that in C. G. S. measure $n = \frac{2\pi}{3600}$. We take $a = 33 \times 10^4$, $g = 981$. Then

$$\frac{\sigma}{\sigma_0} = \frac{1}{290};$$

showing that even for this moderate period the change of sign does not occur until a high degree of rarefaction is reached.

In discarding the restriction to one dimension, we may suppose, without real loss of generality, that $v=0$, and that $u$, $w$, $\rho$, are functions of $x$ and $z$ only. Further we may suppose that $x$ occurs only in the factor $e^{ikx}$; that is, that the motion is periodical with respect to $x$ in the wave-length $\frac{2\pi}{k}$; and that as before $t$ occurs only in the factor $e^{int}$. Equations (5) and (6) then become

$$in\,\rho + iku + \frac{dw}{dz} - \frac{g\,w}{a^2} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

$$a^2\,k\,\rho = -n\,u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$a^2 \frac{d\rho}{dz} = -inw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

from which if we eliminate $u$ and $w$ we get

$$\frac{d^2\rho}{dz^2} - \frac{g}{a^2} \cdot \frac{d\rho}{dz} + \left(\frac{n^2}{a^2} - k^2\right)\rho = 0 \quad . \quad . \quad . \quad . \quad (27)$$

an equation which may be solved in the same form as (12).

One obvious solution of (27) is of importance. If $\frac{d\rho}{dz} = 0$, so that $w=0$, the equations are satisfied by

$$n^2 = k^2\,a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Every horizontal stratum moves alike, and the *proportional* variation of density ($\rho$) is the same at all levels. The possibility of such a motion is evident beforehand, since on account of the assumption of Boyle's law the velocity of sound is the same throughout.

In the application to meteorology, the shortness of the more important periods of the vertical motion suggests that an "equilibrium theory" of this motion may be adequate. For vibrations like those of (28) there is no difficulty in taking account of the earth's curvature. For the motion is that of a simple spherical sheet of air, considered in my book upon the "Theory of Sound," § 333. If $r$ be the radius of the earth, the equation determining the frequency of the vibration corresponding to the harmonic of order $h$ is

$$n^2 r^2 = h\,(h+1)\,a^2 \qquad (29)$$

the actual frequency being $\frac{n}{2\pi}$. If $\tau$ be the period, we have

$$\tau = \frac{2\pi r}{a\sqrt{h\,(h+1)}} \qquad (30)$$

For $h=1$, corresponding to a swaying of the atmosphere from one side of the earth to the opposite,

$$\tau_1 = \frac{2\pi r}{a\sqrt{2}}, \qquad (31)$$

and in like manner for $h=2$.

$$\tau_2 = \frac{2\,\pi r}{\sqrt{6}} = \frac{\tau_1}{\sqrt{3}} \qquad (32)$$

To reduce these results to numbers we may take for the earth's quadrant

$$\frac{1}{2}\pi r = 10^6 \text{ centimeters};$$

and if we take for $a$ the velocity of sound at 0° as ordinarily observed, or as calculated upon Laplace's theory, viz, $33\times 10^3 \frac{\text{centimeters}}{\text{second}}$, we shall find

$$\tau_1 = \frac{4\times 10^9}{\sqrt{2}\times 33\times 10^3} \text{ seconds} = 23.8 \text{ hours}$$

on the same basis,

$$\tau_2 = 13.7 \text{ hours}.$$

It must however be remarked that the suitability of this value of $a$ is very doubtful, and that the suppositions of the present paper are inconsistent with the use of Laplace's correction to Newton's theory of sound propagation. In a more elaborate treatment a difficult question would present itself as to whether the heat and cold developed during atmospheric vibrations could be supposed to remain undissipated. It

is evidently one thing to make this supposition for sonorous vibrations, and another for vibrations of about 24 hours period. If the dissipation were neither very rapid nor very slow in comparison with diurnal changes (and the latter alternative at least seems improbable), the vibrations would be subject to the damping action discussed by Stokes.*

In any case the near approach of $\tau_1$ to 24 hours, and of $\tau_2$ to 12 hours, may well be very important. Beforehand the diurnal variation of the barometer would have been expected to have been much more conspicuous than the semidiurnal. The relative magnitude of the latter, as observed at most parts of the earth's surface, is still a mystery, all the attempted explanations being illusory. It is difficult to see how the operative forces can be mainly semidiurnal in character; and if the effect is so, the readiest explanation would be in a near coincidence between the natural period and 12 hours. According to this view the semidiurnal barometric movement should be the same at the sea level all round the earth, varying (at the equinoxes) merely as the square of the cosine of the latitude, except in consequence of local disturbances due to want of uniformity in the condition of the earth's surface.

TERLING PLACE, WITHAM, *Dec.*, 1889.

---

***Phil. Mag.*, 1851 (4), vol. I, p. 305. Also, Rayleigh: "Theory of Sound," § 247.

## XIX.

# ON THE VIBRATIONS OF AN ATMOSPHERE PERIODICALLY HEATED.*

By MAX MARGULES.

The computation of the variations of pressure in the atmosphere arising from periodic changes in temperature has a certain interest in connection with a problem of meteorology that, like all dynamic problems in this field, necessitates very extensive computations.

The daily variation of the barometer, freed from all non-periodic influences, can be represented very satisfactorily by the super-position of two waves, one of which has a whole day as its period; the other has the half day. The diurnal wave is undoubtedly an effect of the variation of temperature. It appears much stronger on clear days than on cloudy days; it is very slight at sea and shows on the land notable inequalities. The semi diurnal wave is on the other hand of a regularity that is uncommon in meteorological phenomena. At places of the same latitude it is of very nearly equal amplitude and of the same phase in reference to the local time. If we consider this wave also as a consequence of the variations of temperature, then the connection seems to be obscure.

The mean daily temperature represented for any place by a curve, can like every such curve, be analyzed into a series of waves of twenty-four, twelve, eight, and six hour periods. Does the twenty-four-hour wave of pressure originate from the corresponding wave of temperature? Does the twelve-hour variation of pressure depend on the twelve-hour temperature variation? Why is the amplitude of the twelve-hour pressure term so large in comparison with the twenty-four-hour term, whereas the reverse is true for the temperature? Whence come the regularity of the one and the local variations of the other?

These questions have been asked repeatedly. In a memoir recently published,† Hann has given the most comprehensive and thorough description of the daily oscillation of the barometer, utilizing the rich

* Translated from the *Sitzungsberichte der Königlich Akademie der Wissenschaften zu Wien* (Math.), 1890, vol. XCIX, pp. 204–227. See, also, Exner's *Repertorium der Physik*. 1890, Band XXVI, pp. 613–633.

† "Unters. ü. d. tägliche Oscillation d. Barometers," *Vienna Denk.*, vol. 55, 1889.

observational material from all lands and oceans with the object of establishing a basis for a further mathematico-physical theory.

In order to attain this, one must first treat the phenomenon under assumptions that simplify the labor. I believed that some computations as to the variations of pressure in air that is periodically heated would contribute to the better understanding of the diurnal variation of the barometer. In the course of the work, it appeared that the computation must not be confined to the simplest cases if one would make it useful to a certain degree. For this reason the investigation has grown to a larger size than was desired by me.

Before giving the detailed computations let the review of certain results take precedence. Let $T_0$ and $p_0$ indicate the absolute temperature and the pressure of the air when at rest; $T_0(1+\tau)$ and $p_0\,(1+\varepsilon)$ indicate temperature and pressure of air in motion. When $\tau$ is given as a periodic function of the time $t$ and of the locality $x$ then $\varepsilon$ will also appear as such a function.

Let a wave of temperature

$$\tau = A \sin 2\pi\left(\frac{t}{\Theta}+\frac{x}{L}\right)$$

with constant amplitude move in the direction $-x$ in a plane layer of air upon which no other forces are acting.

This will produce a wave of pressure

$$\varepsilon = A\,\frac{L^2}{L^2-c^2\Theta^2}\sin 2\,\pi\left(\frac{t}{\Theta}+\frac{x}{L}\right)$$

where $c$ indicates the velocity of propagation of a free vibration when the process is strictly isothermal; in air at the temperature 273° we have $c=280$ metres per second.

If we assume the length of the wave to equal the circumference of the equator then for a period whose duration is one day and for a pressure $p_0$ expressed as 760 millimetres of the barometer a variation of temperature of one degree will produce a variation of pressure of 4.4 millimetres.

Both temperature and pressure vibrations have the same phases when their velocity of propagation $\left(V=\frac{L}{\Theta}\right)$ is greater than $c$, but opposite phases when it is smaller than $c$. If $V=c$ then will $e$ be indefinitely large, as must occur in the case of a frictionless medium when the forced vibrations have the same period as the free. Again, let a wave of temperature similar to the preceding advance in a plane stratum of air, subject to the influence of constant gravity. The air now moves horizontally in the direction of the progress of the wave and also vertically. The pressure wave on the ground is given by an equation similar to the preceding only in the numerator $c^2\ \Theta^2$ is to be substituted for $L^2$. For the equator, the day and 760 millimetres, a

temperature variation of one degree gives a pressure variation of 1.3 millimetres.

If however the amplitude of the temperature variation is not uniform throughout the whole height, but diminishes with the height so that it diminishes by one-half for each ascent of 1,000 metres, then a temperature variation of ten degrees at the earth's surface gives a variation of pressure at the same level of only 2.4 millimetres.

In respect to the whole-day wave for continental tropical regions one could be satisfied with this result. The agreement, however, is only accidental. The twelve-hour wave of pressure at sea still remains entirely inexplicable. Even on the land one should expect that the amplitudes of the whole-day and half-day waves of pressure would have the same ratio as the corresponding temperature amplitudes, since $\varepsilon$ remains unchanged when we put $\frac{1}{2}$ $L$ and $\frac{1}{2}$ $\Theta$ in place of $L$ and $\Theta$.

The computation would hold good for a cylinder of great diameter equally as for a plane; even under certain restrictions it would also hold good for a mass of air within a circular boundary. But it can only be applied to the atmosphere when the air is divided into a number of zones by vertical walls parallel to the circles of latitude. The zones in the neighborhood of the latitude of 50° would have enormous variations of pressure, and there also two neighboring zones would have opposite phases; the amplitudes diminish thence toward both the pole and the equator.

From the great differences in pressure that are thus obtained for different zones, we see the necessity of reducing to calculation the condition of the air over the whole sphere without any partition walls. I pass over the formulæ for the sphere at rest in order to report upon that part of the computation that apparently offers useful results for the elucidation of the half-day wave of pressure. First, I will present some passages quoted already by Hann from a memoir of Sir William Thomson's.

After speaking of the disproportion between the whole and half day variation of the temperature on one hand and the pressure on the other, Thomson says:* "We must consider the atmosphere as a whole and investigate its vibrations with the help of the formulæ that Laplace has developed for the ocean in the *Mecanique Céleste*, and which, as he has shown, are also applicable to the atmosphere. When in the calculation of the tide-producing force, one introduces the influence of temperature instead of attraction, and develops the oscillations corresponding to the whole day and half day terms of the temperature curve, it will probably be found that in the first case the period of the free oscillations departs more from twenty-four hours than in the second case from twelve hours, wherefore for a relatively small amount of tide-producing force,

* "On the thermo-dynamic acceleration of the earth's rotation," *Proc. R. S. Edinburgh*, 1882, vol. XI. Sir William Thomson. "Mathematical and Physical Papers," London, 1890, vol, III, page 344.

the amplitude of the half-day term will be greater than that of the whole-day term."

This prediction is completely verified. When we execute the computation for an atmosphere considered as a rotating spherical shell in which waves of temperature advance from meridian to meridian according to the equation

$$\tau = C \sin \omega \sin (nt+\lambda)$$

(where $\omega$ = Polar distance, $\lambda$ = geographical longitude, $n$ = velocity of the rotation of the earth), then we find for $T_0=273^0$ the wave of pressure

$$\varepsilon = C \sin (nt+\lambda) \, [1.146 \sin \omega - 0.423 \sin^3 \omega - 0.370 \sin^5 \omega - 0.106 \sin^7 \omega$$
$$-0.018 \sin^9 \omega - 0.002 \sin^{11} \omega - \ . \ . \ . \ ]$$

When however at every place the wave of temperature repeats itself twice daily and we assume

$$\tau = C \sin^2 \omega \sin (2nt+2\lambda)$$

then there results

$$\varepsilon = -C \sin (2nt+2\lambda) \, [37.99 \sin^4 \omega + 23.06 \sin^6 \omega$$
$$+5.75 \sin^8 \omega + 0.81 \sin^{10} \omega + 0.07 \sin^{12} \omega + \ . \ . \ . \ ]$$

The law according to which the amplitude of the temperature wave diminishes from the equator toward the pole has been assumed different in the two cases only because of the easier computation; this however is of slight influence in the general result which is, that for equal variations of temperature the resulting variations of pressure become much greater in the double daily wave than in the single wave. The coefficients of the first sine series vary only very slowly with $T_0$ (or with $n$ when we, as Thomson does, consider the period as the variable). It is otherwise in the half-day wave; here the factor of $\sin^4 \omega$ in the neighborhood of $T_0=268^\circ$ passes, from $-\infty$ over to $+\infty$ precisely as in the plane wave before considered when the velocity of propagation of the forced vibration is made equal to that of the free vibration. Thus slight semi-diurnal waves of temperature of scarcely appreciable amplitude are sufficient to produce great waves of pressure in frictionless air if we assume the temperature of the spherical shell to be in the neighborhood of 268°.

Thus far the computation. Its application to the daily variation of the barometer is only clear as to one point. The semi-diurnal wave of pressure may be considered as a consequence of a semi-diurnal wave of temperature of small amplitude. Thus is explained the relative magnitudes (of the diurnal and semi-diurnal temperature and pressure waves) but not the uniformity of the semi-diurnal variation of pressure over the land and the ocean. This uniformity has led Hann to seek the origin of the phenomenon in the absorption of heat by the upper strata of air. But the lower strata have also a semi-diurnal temperature va-

riation and one that varies with locality and with the condition as to cloudiness. It is a question whether the variations of pressure thence resulting are so small in comparison with the regular variations that they are not very noticeable in the averages.

The neglect of the friction and the vertical motion of the air in our last calculations, the assumption of a constant mean temperature for the whole mass of air, and the assumption that for equal latitudes we have equally large ranges of temperature and pressure, allow us to make only the most general application to the case of nature. A more perfect calculation, taking account of the actual distribution of land and water, would be as difficult to execute as would be the computation of the rise and fall of the tides for an ocean of irregular shape, or even for one bounded by meridians.

### I. MOVEMENT OF THE AIR IN VERTICAL PLANES.

*Notation.* $u$ = horizontal velocity along the axis of $x$; $w$ = vertical velocity positive upwards along the axis of $z$; $\mu$ = density; $p$ = pressure; $T$ = absolute temperature; $t$ = time; $g$ = the acceleration of gravity; $R$ = a constant.

We imagine the earth as an infinite plane above which, in all east-west vertical planes, the movement of the air occurs in a similar manner. For slight velocities that allow us to neglect terms in the equations of motion that are of the second degree in $u$ and $w$, these equations, together with the equations of continuity and of elasticity are as follows:*

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\mu}\frac{\partial p}{\partial x} \\ \frac{\partial w}{\partial t} &= -\frac{1}{\mu}\frac{\partial p}{\partial z} - g \\ \frac{\partial \mu}{\partial t} + \frac{\partial(\mu u)}{\partial x} &+ \frac{\partial(\mu w)}{\partial z} = 0 \\ p &= R\mu T. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

If the atmosphere is at rest then $p$, $\mu$, $T$, have the value $p_0$, $\mu_0$, $T_0$, which are functions of the altitude only,

$$\left.\begin{aligned} \frac{1}{p_0}\frac{d p_0}{dz} &= -\frac{g}{RT_0} \\ \frac{1}{\mu_0}\frac{d\mu_0}{dz} &= -\frac{g}{RT_0} - \frac{1}{T_0}\frac{dT_0}{dz} \\ p_0 &= R\mu_0 T_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2)$$

[* The expression "Zustands-Gleichung der Gase," which is applied in Germany to the equation $p\,v = RT$ has, I believe, no single equivalent in ordinary English scientific phraseology unless we adopt the very inelegant historical title Boyle-Mariotte-Gaylussac-Charles-Law. It is the law connecting density, temperature, volume, or pressure, and expresses the simple fact that the substance is truly gaseous. But the characteristic of a gas is its elasticity, and the equation gives the elastic pressure.—*C. A.*]

The motion is caused by small variations of temperature. Such variations will, as a rule, produce only slight variations of density and of pressure. If we put

$$p = p_0 (1 + \varepsilon)$$
$$\mu = \mu_0 (1 + \sigma)$$
$$T = T^0 (1 + \tau)$$

then $\varepsilon$, $\sigma$, $\tau$, are small numbers whose products and squares we shall neglect.

From the following equations,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= - RT_0 \frac{\partial \varepsilon}{\partial x} \\ \frac{\partial w}{\partial t} &= - RT_0 \frac{\partial \varepsilon}{\partial z} + g\tau \\ \frac{\partial \sigma}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - w\left(\frac{g}{RT_0} + \frac{1}{T_0}\frac{dT_0}{dz}\right) &= 0 \\ \varepsilon &= \sigma + \tau \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which take the place of (1), we eliminate $u$, $w$, $\sigma$, by differentiating the first according to $x$, the second according to $z$, and the third according to $t$.

We thus obtain the following differential equation, in which $\tau$ is to be considered as a given function of $x$, $z$, and $t$, but $\varepsilon$ as a function of $x$, $z$, and $t$ that is still to be determined.

$$\left.\begin{aligned} &\frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial z^2} - \frac{g}{RT_0}\frac{\partial \varepsilon}{\partial z} - \frac{1}{RT_0}\frac{\partial^2 \varepsilon}{\partial t^2} \\ &= \frac{g}{RT_0}\frac{\partial \tau}{\partial z} - \frac{g}{RT_0}\left(\frac{g}{RT_0} + \frac{1}{T_0}\frac{dT_0}{dz}\right)\tau - \frac{1}{RT_0}\frac{\partial^2 \tau}{\partial t^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

Before we treat the equation for motions in two dimensions we will consider the simplest case of linear vibrations.

## II. LINEAR VARIATIONS.

When $g = 0$, and $\tau$ and $\varepsilon$ depend only on $t$ and $x$, equation (4) becomes

$$\frac{\partial^2 \varepsilon}{\partial t^2} - RT_0 \frac{\partial^2 \varepsilon}{\partial x^2} = \frac{\partial^2 \tau}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

and when $\tau = 0$, this becomes the Newtonian equation for acoustic vibrations in the atmosphere which gives $c = \sqrt{RT_0}$ as the velocity of propagation.

If we consider—not the variation of temperature, but the flow of heat as known, then we have to introduce the relation.

$$dQ = C_v\, dT + p\, d\left(\frac{1}{\mu}\right) = C_v\, T_0\, d\tau - RT_0\, d\sigma = C_p\, T_0\, d\tau - RT_0\, d\varepsilon$$

where the change of kinetic energy is omitted, as being a quantity of

the second degree, in $u$; $dQ =$ the heat imparted to the unit mass of air during the time $dt$; $C_v =$ specific heat of air under constant volume; $C_p =$ specific heat under constant pressure

$$C_p = C_v + R$$

$$\tau = \frac{Q}{C_v T_0} + \frac{R}{C_v}\sigma = \frac{Q}{C_p T_0} + \frac{R}{C_p}\varepsilon.$$

By combining this last equation with (4$a$) we obtain

$$\frac{\partial^2 \varepsilon}{\partial t^2} - R T_0 \frac{C_p}{C_v}\frac{\partial^2 \varepsilon}{\partial x^2} = \frac{1}{C_v T_0}\frac{\partial^2 Q}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

which converts into the Laplacian equation when $Q = 0$. In this the temperature variations of the air for rapid acoustic vibrations produced by adiabatic compressions and expansions are considered, and the velocity of propagation is therefore

$$c' = \sqrt{R T_0 \frac{C_p}{C_v}}$$

For our purpose it will be more convenient to consider the pressure variations as a consequence of the temperature variations not as a consequence of the variable flow of heat. We therefore return to equation (4$a$).

### III. WAVE OF TEMPERATURE.

A progressive wave of temperature

$$\tau = A \sin (nt + mx) = A \sin 2\pi \left(\frac{t}{\Theta} + \frac{x}{L}\right) \quad . \quad . \quad . \quad . \quad (5)$$

causes a wave of pressure

$$\left.\begin{aligned} \varepsilon &= B \sin (nt + mx) \\ B &= \frac{L^2}{L^2 - c^2\Theta^2} A \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

advancing in the same direction.

$\frac{L}{\Theta} = V$ is the velocity of the progress of both of these waves. The phases of the waves are the same or opposite according as $V$ is larger or smaller than $c$. But $V = c$ leads to an infinitely large value of $B$, a result to which we must always come when in a frictionless medium the period of the forced vibrations agrees with those of the free.

For the atmosphere we have

$$R = \frac{10333 \times 9.806}{273 \times 1.293} = 287.0.$$

Here, and in the following, we adopt as units the metre, the kilogram, the second of time, the degree of the Centigrade thermometer, and for pressures the barometric scale. For $T_0=273°$ we have $c=279.9$.

The values $L=4\times10^7$ or the circumference of the equator, $\Theta=24\times60\times60$ or 1 day and $T_0=273°$ gives a wave of pressure whose maximum coincides with the maximum of temperature, and also gives $B=1.576\times A$. A temperature variation of 1° C. produces a pressure variation $\frac{p_0\times1.576}{273}$ or 4.4 millimetres of mercury when $p_0$ is 760 on the barometer scale.

When we desire to obtain pure horizontal vibrations in a layer of appreciable altitude without neglecting force of gravity we should have to introduce a function ($A$) of the altitude as we see from the equations (3), that shall satisfy the condition

$$\frac{1}{A}\frac{dA}{dz}=\frac{g}{c^2}\frac{L^2-c^2\Theta^2}{L^2}$$

For isothermal vibrations in a vertical column of air the conditions are

$$\tau=0$$

$$\frac{\partial\varepsilon}{\partial x}=0$$

and equation (4) becomes

$$\frac{\partial^2\varepsilon}{\partial z^2}-\frac{g}{RT_0}\frac{\partial\varepsilon}{\partial z}-\frac{1}{RT_0}\frac{\partial^2\varepsilon}{\partial t^2}=0.$$

This equation or the corresponding equation in $w$ has recently been discussed at length by Lord Rayleigh (*Phil. Mag.*, Feb., 1890).*

## IV. VIBRATIONS OF THE AIR WHEN A WAVE OF TEMPERATURE ADVANCES HORIZONTALLY, TAKING INTO CONSIDERATION THE FORCE OF GRAVITY.

With a constant value of $T_0$ and putting $\tau=A\sin(mx+nt)$ the differential equation (4) becomes

$$\frac{\partial^2\varepsilon}{\partial x^2}+\frac{\partial^2\varepsilon}{\partial t^2}-\alpha\frac{\partial\varepsilon}{\partial z}-\frac{\alpha}{g}\frac{\partial^2\varepsilon}{\partial t^2}=-\frac{\alpha}{g}\frac{\partial^2\tau}{\partial t^2}-\alpha^2\tau \quad . \quad . \quad . \quad . \quad (4c)$$

$$\left[\alpha=\frac{g}{RT_0}\right]$$

The wave of pressure will be of the form $\varepsilon=F(z)\sin(mx+nt)$.

* [See also No. XVIII of this present collection of Translations.]

The notation and solution are as follows :

$$\frac{d^2F}{dz^2}-\alpha\frac{dF}{dz}+hF=\left(\frac{\alpha}{g}n^2-\alpha^2\right)A$$

$$\left[h=\frac{\alpha}{g}n^2-m^2\right]$$

$$F(z)=B+K_1e^{k_1z}+K_2e^{k_2z}$$

$$B=\frac{A}{h}\left(\frac{\alpha}{g}n^2-\alpha^2\right)$$

$$k_1=\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}-h} \qquad k_2=\frac{\alpha}{2}+\sqrt{\frac{\alpha_2}{4}-h}$$

In order to determine the constants of integration $K_1$ and $K_2$ whose factors in the expression for $\varepsilon$ represent free vibrations we note that $w=0$ when $z=0$ and also when $z$ has a very large value$=Z$ which corresponds to a fictitious upper plane bounding the atmosphere. From the second of equations (3) we obtain

$$w=\frac{g}{\alpha n}\left(K_1k_1e^{k_1z}+K_2k_2e^{k_2z}-\alpha A\right)\cos(mx+nt)$$

The boundary conditions give

$$K_1k_1+K_2k_2=\alpha A$$

$$K_1k_1e^{k_1Z}+K_2k_2e^{k_2Z}=\alpha A$$

$$K_1k_1=\alpha A\frac{e^{k_2Z}-1}{e^{k_2Z}-e^{k_1Z}}$$

$$K_2k_2=\alpha A\frac{1-e^{k_1Z}}{e^{k_2Z}-e^{k_1Z}}$$

If now, as in our example (where the wave length is the circumference of the earth and the period is one day), $h$ is very small compared with $\alpha^2$, then is $k$ very small, and $k_2$ nearly equal to $\alpha$. Hence, $K_2$ will be smaller in proportion as $Z$ is larger. If we desire to apply the resulting formula only to altitudes that are slight in comparison with $Z$, then will $K_2e^{k_2z}$. With this limitation we put $K_2=0$ and $K_1k_1=\alpha A$, and obtain

$$w=A\frac{g}{n}\left(e^{k_1z}-1\right)\cos(mx+nt)$$

$$\varepsilon=A\left(\frac{\alpha n^2}{gh}-\frac{\alpha^2}{h}+\frac{\alpha}{k_1}e^{k_1z}\right)\sin(mx+nt)$$

Under the assumption that $\frac{h}{\alpha^2}$ is a small quantity we have

$$k_1=\alpha\left(\frac{h}{\alpha^2}+\frac{h^2}{\alpha^4}\right)$$

$$\frac{\alpha}{k_1}=\frac{\alpha^2}{h}-1,$$

and when we retain only the first two terms of the exponential series we obtain

$$\varepsilon=A\left(\frac{m^2}{h}+\alpha z\right)\sin\,(mx+nt)=A\left(\frac{c^2\Theta^2}{L^2-c^2\Theta^2}+\alpha z\right)\sin\,2\pi\left(\frac{t}{\Theta}+\frac{x}{L}\right).$$

For $L=4\times10^7$, $\Theta=24\times60\times60$, we obtain

$$\varepsilon=A\,(0.576+0.000125z)\sin\,(mx+nt).$$

The relative variations of pressure near the earth's surface increase very slowly with the altitude. At the surface of the earth itself the variations of pressure are appreciably smaller in the ratio of $\frac{0.576}{1.576}$ than in the example of the third section, where purely horizontal vibrations occurred. A daily variation of temperature of 1° C. would in the present case cause a pressure variation of 1.6$^{mm}$. The phases of both vibrations occur simultaneously when $L>c\Theta$.

## V. A SIMILAR COMPUTATION FOR THE CASE WHEN THE AMPLITUDE OF THE TEMPERATURE VIBRATION DIMINISHES WITH THE ALTITUDE.

The differential equation (4) becomes

$$\frac{\partial^2\varepsilon}{\partial x^2}+\frac{\partial^2\varepsilon}{\partial z^2}-\alpha\frac{\partial\varepsilon}{\partial z}-\frac{\alpha}{g}\frac{\partial^2\varepsilon}{\partial t^2}=\alpha\frac{\partial\tau}{\partial z}-\frac{\alpha}{g}\frac{\partial^2\tau}{\partial t^2}-\alpha^2\tau\quad\ldots\quad(4d)$$

To the assumption $\tau=Ae^{-sz}\sin\,(mx+nt)$ there corresponds

$$\varepsilon=(Be^{-sz}+Ke^{kz})\sin\,(mx+nt)$$

$$B\,(s^2+\alpha s+h)=A\left(\frac{\alpha n^2}{g}-\alpha^2-\alpha s\right)$$

$$k=\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}-h}$$

$h$ has the same meaning as before. $K$ stands for $K_1$ and $K_2$ disappears under the same limitations as before, (namely, that the result is to be applied only to altitudes that are slight in comparison to $Z$).

From the condition $w=0$ when $z=0$, there follows $Kk=Bs+A\alpha$, hence

$$\varepsilon=\frac{A}{s^2+\alpha s+h}\left\{\left(\frac{\alpha}{g}n^2-\alpha^2-\alpha s\right)e^{-sz}+\frac{\alpha}{k}\left(\frac{n^2}{g}s+h\right)e^{kz}\right\}\sin(mx+nt)$$

If $\frac{h}{\alpha^2}$ is very small, and $s$ of the same order of magnitude as $\alpha$, or even much larger, then for values of $z$ that are not too large, this last equation becomes

$$\varepsilon=A\left(\frac{\alpha}{s+\alpha}\frac{m^2}{h}+\alpha z\right)\sin(mx+nt)$$

$$=A\left(\frac{\alpha}{s+\alpha}\frac{c^2\,\Theta^2}{L^2-c^2\,\Theta^2}+\alpha z\right)\sin 2n\left(\frac{x}{L}+\frac{t}{\Theta}\right)$$

If we put $s=0.000693$, then, at an altitude of 1,000 metres, the variation of temperature will be half as large as at the surface of the earth. With this value, and the same values of $L$ and $\Theta$ as above, there results

$$\varepsilon=A\,(0.153\times 0.576+0.000125z)\sin(mx+nt)$$

Hence, for a mean temperature of 273°, a barometric variation of 2.45 millimetres is produced by a daily variation of 10° in temperature at the surface of the earth.

## VI. TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS FOR SPHERICAL COÖRDINATES.

Instead of the rectilinear coördinates $x$, $y$, $z$, the spherical coördinates ($r=$ radius; $\omega=$ polar distance; $\lambda=$ east longitude from adopted meridian), are to be introduced

$$x=r\sin\omega\cos\lambda,$$
$$y=r\sin\omega\sin\lambda,$$
$$z=r\cos\omega.$$

The equations of motion of a point on which the forces $X$, $Y$, $Z$ are acting along the rectilinear axes, which are $X=\frac{d^2x}{dt^2}$, etc., are thus transformed into the following:

$$\left.\begin{aligned}P&=\frac{d^2r}{dt^2}-r\left(\frac{d\omega}{dt}\right)^2-r\sin^2\omega\left(\frac{d\lambda}{dt}\right)^2\\ \Omega&=r\frac{d^2\omega}{dt^2}+2\frac{dr}{dt}\frac{d\omega}{dt}-r\cos\omega\sin\omega\left(\frac{d\lambda}{dt}\right)^2\\ \Lambda&=r\sin\omega\frac{d^2\lambda}{dt^2}+2\sin\omega\frac{dr}{dt}\frac{d\lambda}{dt}+2r\cos\omega\frac{d\omega}{dt}\frac{d\lambda}{dt}\end{aligned}\right\}\quad\ldots\quad(7)$$

where $P$, $\Omega$, $\Lambda$ are the components of the forces in the directions of the new coördinates $dr$, $r\,d\omega$, $r \sin \omega\, d\,\lambda$. If the velocities are so small that we can neglect their squares and products, then only the first term will remain on the right-hand side of each of these equations. If we put

$$\frac{dr}{dt}=a, \qquad r\frac{d\omega}{dt}=b, \qquad r \sin \omega \frac{d\lambda}{dt}=c$$

we have

$$P=\frac{da}{dt}, \qquad \Omega=\frac{db}{dt}, \qquad \Lambda=\frac{dc}{dt}.$$

Therefore the equations of motion of a fluid that is only under the influence of a constant force of gravity positive in the direction of the diminishing radius, are

$$\left.\begin{aligned} -g-\frac{1}{\mu}\frac{\partial p}{\partial r}&=\frac{\partial a}{\partial t} \\ -\frac{1}{\mu}\frac{\partial p}{r\partial\omega}&=\frac{\partial b}{\partial t} \\ -\frac{1}{\mu}\frac{\partial p}{r \sin \omega\, \partial\, \lambda}&=\frac{\partial c}{\partial t} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

These equations are applicable to the motion on a sphere at rest. In order to investigate the relative motion on the rotating terrestrial sphere, we modify equation (7) in that we put $\nu t+\lambda$ in place of $\lambda$ where $\nu$ is the velocity of rotation of the earth. In place of $\frac{d\lambda}{dt}$ in equation (7) there now occurs $\frac{\nu+d\lambda}{dt}$. If, again, we put $c$ in place of the new $r \sin \omega\frac{d\lambda}{dt}$, if we retain the products $\nu a$, $\nu b$, $\nu c$, and if on the other hand we omit the terms in $\nu^2$, which indicate only a slight change in the force of gravity, then we obtain the equations for the motion of a fluid on a rotating sphere. On the right-hand sides of the equations (8) the terms $-2\nu c \sin \omega$, $-2\nu c \cos \omega$ and $+2\nu a \sin \omega+2\nu b \cos \omega$ are to be added respectively.

The equation of continuity has the same form for the sphere at rest as for the rotating sphere.

$$\frac{\partial\mu}{\partial t}+\frac{\partial(\mu r^2 a)}{r^2\partial r}+\frac{\partial(\mu\, b \sin \omega)}{r \sin \omega\, \partial\omega}+\frac{\partial(\mu c)}{r \sin \omega\, \partial\, \lambda}=0 \quad \ldots\ldots \quad (9)$$

Introducing the notation

$$p=p_0(1+\varepsilon), \qquad T=T_0(1+\tau)$$

allied to that above used, we obtain the following differential equations

for the motion of the atmosphere on the rotating sphere that result from small variations of the temperature $\tau$

$$\left.\begin{aligned} g\tau - RT_0\frac{\partial\varepsilon}{\partial r} &= \frac{\partial a}{\partial t} - 2\nu\, c \sin\omega \\ -RT_0\frac{\partial\varepsilon}{r\partial\omega} &= \frac{\partial b}{\partial t} - 2\nu\, c\cos\omega \\ -RT_0\frac{\partial\varepsilon}{r\sin\omega\,\partial\lambda} &= \frac{\partial c}{\partial t} + 2\nu\, a\sin\omega + 2\nu\, b\cos\omega \\ \frac{\partial\varepsilon}{\partial t} - \frac{\partial\tau}{\partial t} + \left(\frac{2}{r} - \frac{g}{RT_0}\right) a + \frac{\partial a}{\partial r} + \frac{\partial(b\sin\omega)}{r\sin\omega\partial\omega} &+ \frac{\partial c}{r\sin\omega\partial\lambda} = 0 \end{aligned}\right\} \quad (10)$$

If $\nu=0$, these give the corresponding equations for the sphere at rest.

## VII. THE ATMOSPHERE WITHIN A SPHERICAL SHELL AT REST.

As in the first computation in the second section for the case of a plane we shall assume only horizontal motions. Moreover the radius of the sphere $S$ will be assumed very large in proportion to the height of the stratum of air. If in equation (10) we substitute $S$ instead of $r$, put $a=0$ and $\nu=0$ and eliminate $b$ and $c$ from the last three equations, there results

$$\frac{S^2}{RT_0}\left(\frac{\partial^2\tau}{\partial t^2} - \frac{\partial^2\varepsilon}{\partial t^2}\right) + \frac{1}{\sin\omega}\frac{\partial}{\partial\omega}\left(\frac{\partial\varepsilon}{\partial\omega}\sin\omega\right) + \frac{\partial^2\varepsilon}{\sin^2\omega\,\partial\lambda^2} = 0 \quad . \quad (11)$$

*Single daily wave.* The wave of temperature

$$\tau = A \sin\omega \sin(nt + \lambda)$$

causes a wave of pressure

$$\varepsilon = B \sin\omega \sin(nt + \lambda)$$

where $A$ and $B$ have the relation

$$B\left(\frac{n^2 S^2}{R\,T_0} - 2\right) = A\,\frac{n^2 S^2}{R\,T_0}$$

With $T_0 = 273°$, $n = \dfrac{2\pi}{24 \times 60 \times 60}$, $S$ = radius of the earth, and $p_0$ = 760 *mm.*, a variation of temperature of 1° on the equator will produce a variation of pressure at the equator of 10.4 *mm.* $B$ will be equally large for the spherical shell as for a plane wave of the same periodic time, when we assume the wave length for the plane to be equal to the circumference of the circle of 45° latitude on the sphere.

*Double daily wave.* For the temperature wave

$$\tau = A \sin^2\omega \sin(2\,nt + 2\,\lambda)$$

we obtain the pressure wave

$$\varepsilon = B \sin^2\omega \sin(2\,nt + 2\,\lambda)$$

with the following relation between $A$ and $B$

$$B\left(\frac{4\,n^2\,S^2}{R\,T_0}-6\right)=A\frac{4\,n^2\,S^2}{R\,T_o}$$

With the same constants as before 1° variation of temperature on the equator gives 6.2 *mm.* variation of pressure.

On the occasion of the computation for the rotating sphere we shall again have opportunity to explain that the particular integrals that we, in both cases, have given as the solution of the differential equation (11) contain the complete solution for the whole spherical shell.

If we put $\Theta_1$ for the duration of the vibration for single waves for which $B$ is infinitely large, and similarily $\Theta_2$ for the double wave, then we have

$$\Theta_1=\frac{2\,\pi}{n_1}=\frac{2\,\pi\,S}{\sqrt{2\,R\,T_0}}$$

$$\Theta=\frac{2\,\pi}{2\,n_2}=\frac{2\,\pi\,S}{\sqrt{6\,R\,T_0}}$$

These are the values of the periods of free vibrations of a spherical shell. Lord Rayleigh (*L. E. D. Phil. Mag.* Feb. 1890) investigates only such and finds (by putting $\sqrt{R\,T_0\,\frac{C_p}{C_v}}$ for the velocity of propagation instead of $\sqrt{R\,T_0}$) for the atmosphere on the earth at rest $\Theta_1=23.8$ hours and $\Theta_2=13.7$ hours; therefore the first is much nearer to 24 than the second is to 12 hours. He remarks however that it is doubtful whether one ought to adopt the Laplacian velocity of propagation for vibration of such long duration.

Therefore the relative magnitudes of the semi-diurnal variation of the barometer still remains a riddle. But this is so only so long as we confine the calculations to the sphere at rest.

## VIII. CALCULATION FOR A ROTATING SPHERE.

*Diurnal wave.*—In this case also the calculation will be carried out only for air in a spherical shell whose thickness is small in comparison with the radius $S$ of the sphere, and also under the further assumption that the movements are horizontal, and that therefore $a=0$. [This latter assumption and the omission of the first of equations (10) are certainly not unobjectionable; they are imitated from the analogous processes in the theory of the tides.] The difference between the sidereal day and the solar day is not considered, and $\nu=n$

$$\left.\begin{aligned}&-\frac{R\,T_0}{S}\frac{\partial\varepsilon}{\partial\omega}=\frac{\partial b}{\partial t}-2nc\cos\omega\\&-\frac{R\,T_0}{S}\frac{\partial\varepsilon}{\sin\omega\partial\lambda}=\frac{\partial c}{\partial t}+2nb\cos\omega\\&S\left(\frac{\partial\varepsilon}{\partial t}-\frac{\partial\tau}{\partial t}\right)+\frac{1}{\sin\omega}\left\{\frac{\partial(b\sin\omega)}{\partial\omega}+\frac{\partial c}{\partial\lambda}\right\}=0\end{aligned}\right\}\quad\ldots\quad(10a)$$

When $\tau=A\,(\omega)\sin(nt+\lambda)$, then $\varepsilon$, $b$, $c$, are to be sought in expressions of the following form:

$$\varepsilon=E\,(\omega)\sin(nt+\lambda),$$
$$b=\varphi\,(\omega)\cos(nt+\lambda),$$
$$c=\psi\,(\omega)\sin(nt+\lambda),$$

wherefore the last of equations (10$a$) becomes

$$nS\,(E-A)+\frac{1}{\sin\omega}\left\{\frac{d(\varphi\sin\omega)}{d\omega}+\psi\right\}=0,$$

whilst the first two give

$$\varphi=\frac{RT_0}{nS}\,\frac{\dfrac{dE}{d\omega}+E\dfrac{2\cos\omega}{\sin\omega}}{1-4\cos^2\omega}$$

$$\psi=-\frac{RT_0}{nS}\,\frac{\dfrac{dE}{d\omega}2\cos\omega+\dfrac{E}{\sin\omega}}{1-4\cos^2\omega}$$

These latter values substituted in the preceding equation lead to a relation between $E$ and $A$ only, or between $\varepsilon$ and $\tau$. It will be convenient for the further computation to introduce an auxiliary function, $\Phi\,(\omega)$;

$$\left.\begin{aligned}
&\Phi\,(\omega)=\frac{nS}{RT_0}\varphi\,(\omega)\sin(\omega)\\
&(1-4\cos^2\omega)\,\Phi\omega=\frac{1}{\sin\omega}\,\frac{d\,(E\sin^2\omega)}{d\omega}\\
&E=\frac{1}{\sin^2\omega}\int\Phi\,(\omega)\sin\omega\,(4\sin^2\omega-3)\,d\omega\\
&\frac{n^2S^2}{RT_0}(E-A)+\frac{1}{\sin\omega}\left\{\frac{d\Phi}{d\omega}-\Phi\frac{2\cos\omega}{\sin\omega}-\frac{E}{\sin\omega}\right\}=0
\end{aligned}\right\}\quad .\ .\ (11.)$$

If we assume $\Phi$ to have the following form:

$$\Phi\,(\omega)=\cos\omega\,(a_1\sin\omega+a_3\sin^3\omega+a_5\sin^5\omega+\quad .\ .\ .\ )$$

then

$$E\,(\omega)=b_1\sin\omega+b_3\sin^3\omega+b_5\sin^5\omega+\quad .\ .\ .$$

$$b_1=a_1,\ b_3=\frac{4a_1-3a_3}{5},\ b_5=\frac{4a_3-3a_5}{7},\quad .\ .\ .$$

Let the temperature amplitude diminish from the equator to the pole according to the cosine of the latitude or

$$A\,(\omega)=C\sin(\omega)$$

and for brevity put

$$k=n^2\frac{S^2}{RT_0}$$

then we obtain the following equations for the determination of the constants

$$\left.\begin{aligned}
&\left(1+\frac{3}{5}\right)a_3-\left(k+\frac{4}{5}\right)a_1-k\,C=0\\
&\left(3+\frac{3}{7}\right)a_5-\left(\frac{3}{5}k+\frac{4}{7}+2\right)a_3+\frac{4}{5}\,k\,a_1=0\\
&\left(i-2+\frac{3}{i+2}\right)a_i-\left(\frac{3}{i}k+\frac{4}{i+2}+i-3\right)a_{i-2}+\frac{4}{i}\,k\,a_{i-4}=0\\
&\qquad\qquad i=5,\ 7,\ 9,\ \ldots
\end{aligned}\right\}\ldots(11a)$$

Apparently $a_1$ remains undetermined; for the computation of the others, following the lead of Laplace, we write

$$\frac{a_{i-2}}{a_{i-4}}=\frac{4\,k\,(i+2)}{3\,k\,(i+2)+(i-2)\,i\,(i+2)-(i-1)\,i\,(i+1)\frac{a_i}{a_{i-2}}}$$

By the interchange of $i$ with $i+2$ a similar expression is formed for $\frac{a_i}{a_{i-2}}$ and then $\frac{a_{i+2}}{a_i}$ and in a similar manner for the subsequent terms of the series, and by substituting these values in the above equation we obtain a continued rapidly converging fraction.

$$q_3=\frac{a_5}{a_3}=\frac{4\,k\,9}{N_3-\frac{Z_5}{N_5-\frac{Z_7}{N_7-\ldots}}}$$

$$N_1=3\,k.\,7+3.\,5.\,6,$$

$$N_3=3\,k.\,9+5.\,7.\,8,$$

$$Z_3=4\,k.\,4.\,5.\,6.\,9$$

$$q_5=\frac{a_7}{a_5}=\frac{4\,k\,11}{N_5-\frac{Z_7}{N_7-\frac{Z_9}{N_9-\ldots}}}$$

$$N_5=3\,k.\,11+7.\,9.\,10,$$

$$Z_5=4\,k.\,6.\,7.\,8.\,11,$$

If in the second of equations (11$a$) we put $a_5=q_3\,a_3$, then will $\frac{a_3}{a_1}$ also be determined, and the quotient has the same value as if it were computed from the serial fraction

$$q_1=\frac{a_3}{a_1}=\frac{4\,k.\,7}{N_1-\frac{Z_3}{N_3-\frac{Z_5}{N_5-\ldots}}}$$

By the first of equations (11$a$) we obtain also the value of $a_1$; consequently that of

$$a_3 = q_1\, a_1$$
$$a_5 = q_1\, q_3\, a_1, \text{ etc.}$$

If, in the computation of $q_1$ we take a sufficient number of fractions, as, for instance, up to $N_{19}$, we have thereby also performed the greater part of the numerical computation for $q_3$, $q_5$, and $q_7$.

This remarkable method of determining the constants was by Laplace applied to the theory of the tides. Its true importance was first recognized again by Sir William Thomson, who defended it against Airy.* Without Thomson's commentary the copy would not be easy to understand. In our case the matter presents itself very similarly. The differential equation (11), when we replace $\varphi$ by $E$, is of the second order, and should have an integral with two arbitrary constants. These can be determined when on two arbitrary circles of latitude, certain conditions are to be fulfilled, such for instance as $\varepsilon = 0$, or $b = 0$. One constant drops out when we let one of the parallel circles coincide with the pole; the other is in this case to be determined as if the second parallel was the equator itself. At the equator, on account of the symmetry, we must have $b = 0$. The equatorial plane is to be considered as a fixed partition.

The computation assumes that $\dfrac{a_i}{a_{i-2}}$ converges towards 0 as $i$ increases. If we assume for $a_1$ not the value that results from the computation of the continued fraction but some other arbitrary one, and therewith compute $a_3$, $a_5$, etc., by equation (11a), we obtain a series that diverges for the equator, where $\sin\omega = 1$.

I have computed the constants with two values of $k$. First,

$$k = 2.5 \qquad S = \frac{4 \times 10^7}{2\pi} \qquad R = 287.0$$
$$n = \frac{2\pi}{24 \times 60 \times 60} \qquad T_0 = 298.7^\circ$$

And second, for

$$k = 2.7352 \qquad T_0 = 273^\circ$$

If we also write $\left\{\begin{array}{l} \alpha_1 C \text{ instead of } a_1, \\ \alpha_3 C \text{ instead of } a_3, \\ \beta_1 C \text{ instead of } b_1, \\ \beta_3 C \text{ instead of } b_3, \end{array}\right.$

We find—

$$\left.\begin{array}{l} \tau = C \sin\omega\,(nt + \lambda), \\ \Phi = C \cos\omega\,(\alpha_1 \sin\omega + \alpha_3 \sin^3\omega + \ \ .\ \ .\ \ .) \\ \varepsilon = C \sin(nt + \lambda)\,[\beta_1 \sin\omega + \beta_3 \sin^3\omega + \beta_5 \sin^5\omega + \ \ .\ \ .\ \ .] \end{array}\right\} \quad (12)$$

* Airy; "On an Alleged Error in Laplace's Theory of Tides." *Phil. Mag.*, 1875 (4), vol. L., p. 227.

| | $\alpha_1$ | $\alpha_3$ | $\alpha_5$ | $\alpha_7$ | $\alpha_9$ |
|---|---|---|---|---|---|
| $k = 2.5$ | $-1.119$ | $-0.745$ | $-0.232$ | $-0.040$ | $-0.004$ |
| $k = 2.7352$ | $-1.146$ | $-0.823$ | $-0.279$ | $-0.053$ | $-0.006$ |

| | $\beta_1$ | $\beta_3$ | $\beta_5$ | $\beta_7$ | $\beta_9$ |
|---|---|---|---|---|---|
| $k = 2.5$ | $1.119$ | $-0.448$ | $-0.326$ | $-0.090$ | $-0.013$ |
| $k = 2.7352$ | $1.146$ | $-0.423$ | $-0.370$ | $-0.106$ | $-0.018$ |

With the value of $k = 2.7352$ we obtain as the sum of the series of sines within the [ ] in the value of $\varepsilon$:

| | |
|---|---|
| On the equator . . . . . . | 0.23 |
| At latitude 30° . . . . . . | 0.50 |
| At latitude 45° . . . . . . | 0.58 |
| At latitude 60° . . . . . . | 0.51 |

Therefore the variation of pressure has a maximum in the neighborhood of 45° when we assume the variation of temperature to be proportional to the cosine of the latitude. For $2C = \frac{1}{273}$, *i. e.*, for a variation of temperature of 1° at the equator there results a variation of pressure of 0.64 millimetres at the equator, but 1.6 millimetres at latitude 45°.

In order to investigate how the result is affected when we assume that the temperature amplitude diminishes more rapidly from the equator to the pole, we will carry out the computation for still another case, namely—

$$A(\omega) = C \sin^3 \omega,$$

which gives for the determination of $a$ the equations—

$$\left(1 + \frac{3}{5}\right) a_3 - \left(k + \frac{4}{5}\right) a_1 = 0.$$

$$\left(3 + \frac{3}{7}\right) a_5 - \left(2 + \frac{4}{7} + \frac{3}{5} k\right) a_3 + \frac{4}{5} k a_1 = kC.$$

$$\left(5 + \frac{3}{9}\right) a_7 - \left(4 + \frac{4}{9} + \frac{3}{7} k\right) a_5 + \frac{4}{7} k a_3 = 0. \qquad (11b)$$

The ratio $\frac{a_3}{a_1}$ is given from the first equation, but $q_3$, $q_5$, etc., retain the same values as before. The second equation determines the value of $a_1$. As before we have—

$$\left.\begin{aligned} \tau &= C \sin^3 \omega \sin (nt + \lambda) \\ \varepsilon &= C \sin (nt + \lambda) \left[\beta_1 \sin \omega + \beta_3 \sin^3 \omega + \beta_5 \sin^5 \omega + \ldots\right] \end{aligned}\right\} \quad (12b)$$

For $k = 2.7352$ we have—

| | |
|---|---|
| $\beta_1 = 0.601$ | $\beta_7 = -0.172$ |
| $\beta_3 = 0.316$ | $\beta_9 = -0.030$ |
| $\beta_5 = -0.566$ | $\beta_{11} = -0.003$ |

The sum of the series of sines in the value of $\varepsilon$ is—

| | |
|---|---|
| For the equator . . . . . . | 0.15 |
| For latitude 30° . . . . . . | 0.38 |
| For latitude 45° . . . . . . | 0.42 |
| For latitude 60° . . . . . . | 0.32 |

Again we find a minimum at the equator; the maximum of the pressure amplitude lies between latitudes 30° and 45°; the diminution in the higher latitudes is greater than in the previous examples, but still slow in comparison with the diminution of the temperature amplitude. According to equations (12) and (12*b*) the greatest pressure and highest temperature occur simultaneously.

## IX. ROTATING SPHERE: SEMI-DIURNAL WAVE.

If in the differential equations (10*a*), for the horizontal motions on a rotating sphere, we put

$$\begin{aligned} \tau &= A\,(\omega)\sin\,(2nt+2\lambda) \\ \varepsilon &= E\,(\omega)\sin\,(2nt+2\lambda) \\ b &= \varphi\,(\omega)\cos\,(2nt+2\lambda) \\ c &= \psi\,(\omega)\sin\,(2nt+2\lambda) \end{aligned}$$

there results:

$$\varphi = \frac{RT_0}{2nS}\,\frac{\dfrac{dE}{d\omega}+E\dfrac{2\cos\omega}{\sin\omega}}{\sin^2\omega}$$

$$\psi = -\frac{RT_0}{2nS}\,\frac{\dfrac{dE}{d\omega}\cos\omega+\dfrac{2E}{\sin\omega}}{\sin^2\omega}$$

$$2nS(E-A)+\frac{1}{\sin\omega}\left\{\frac{d(\varphi\sin\omega)}{d\omega}+2\psi\right\}=0$$

After the elimination of $\varphi$ and $\psi$, and when we again put $k=\dfrac{n^2S^2}{RT_0}$ there remains

$$\frac{d^2E}{d\omega^2}\sin^2\omega-\frac{dE}{d\omega}\sin\omega\cos\omega+E(4k\sin^4\omega+2\sin^2\omega-8)=4kA(\omega)\sin^4\omega. \quad (13)$$

If we assume that $A(\omega)=C\sin^2\omega$, we have then to do with the same problem as in the computation of the semidiurnal tide in an ocean of constant depth. Assuming

$$E(\omega)=a_0+a_2\sin^2\omega+a_4\sin^4\omega+a_6\sin^6\omega+\;.\;.\;.\;.\;.$$

there results

$$a_0=0,\ a_2=0,\ a_4 \text{ apparently undetermined,}$$

$$\left.\begin{aligned} (4\times6-8)a_6-(3\times4-2)a_4-4kC&=0 \\ (i^2+6i)a_{i+4}-(i^2+3i)a_{i+2}+4ka_i&=0 \\ i=4,\,6,\,8\;\;.\;\;.\;\;.& \end{aligned}\right\}\quad .\;.\;.\;.\;.\quad (13a)$$

$$q_i=\frac{a_{i+2}}{a_i}=\frac{4k}{i(i+3)-i(i+6)\dfrac{a_{i+4}}{a_{i+2}}}$$

From this we develop the continued fraction as before, and compute the ratios of the constants. But $\alpha_4$ is not now indeterminate, but its value is immediately found to be $-Cq_2$; hence [see Ferrel, p. 320]

$$\alpha_6 = -Cq_2q_4$$
$$\alpha_8 = -Cq_2q_4q_6 \quad . \quad . \quad .$$

$$\left.\begin{array}{l}\tau = C\sin^2\omega\sin(2nt+2\lambda) \\ \varepsilon = C\sin(2nt+2\lambda)[\alpha_4\sin^4\omega + \alpha_6\sin^6\omega + \alpha_8\sin^8\omega + . \ .]\end{array}\right\} \quad . \quad (14)$$

For $4k = 40, =10, =5$, Laplace has computed the constants. Only the middle value of these is of interest for our problem. I have in addition executed the computation for some neighboring values of $k$.

| $4k =$ | 10. | 10.94 | 11. | 11.1 | 11.2 | 12. |
|---|---|---|---|---|---|---|
| $T_0=$ | 298°.7 | 273°.0 | 271°.5 | 269°.1 | 266°.7 | 248°.9 |
| $\alpha_4$ | −6.196 | −37.99 | −55.00 | −247.8 | 101.8 | 8.270 |
| $\alpha_6$ | −3.247 | −23.06 | −33.68 | −154.2 | 64.3 | 5.919 |
| $\alpha_8$ | −0.724 | − 5.75 | − 8.46 | − 39.2 | 16.5 | 1.662 |
| $\alpha_{10}$ | −0.092 | − 0.81 | − 1.20 | − 5.6 | 2.4 | 0.260 |
| $\alpha_{12}$ | −0.008 | − 0.07 | − 0.11 | − 0.5 | 0.2 | 0.026 |

These numbers confirm Thomson's expectations, that the period of the free vibrations of this kind, for a rotating spherical atmosphere of ordinary temperature, lies very near 12 hours. Instead of so determining the velocity of rotation of the earth that the period shall agree exactly with a half-day, we can choose a corresponding temperature. It lies near to 268°. At this point $\alpha_4$ passes from $-\infty$ over to $+\infty$. In the neighborhood of this value forced vibrations must lead to enormously great amplitudes. Therefore a slight semi-diurnal wave of temperature would suffice to produce a very great wave of pressure of the same period. At temperatures below 268° the phases of both are in agreement; in other cases they are opposed.

For $4k=10$, or $T_0=298.°7$, we obtain at the equator

$$\varepsilon = -10.26\ C\sin(2nt+2\lambda)$$

Therefore, a temperature amplitude of $0.038° = \dfrac{298.7}{760 \times 10.26}$ would suffice in order to produce a pressure amplitude of 1 *mm*.

The comparison of the atmosphere with a spherical shell having a constant temperature of 298°.7 gives, as we shall see, the lunar tide on the equator much larger than it is, as deduced from observations. Similarly one must require corresponding large temperature amplitudes in order to produce the observed semidiurnal pressure amplitude of 1 *mm* at the equator. In view of the great imperfections in the assumptions no importance can be attached to the numerical values.

This computation only shows that in order to produce semidiurnal variations of pressure of the same amount as the diurnal variation much

## X. TIDAL EBB AND FLOW OF THE ATMOSPHERE.

In order to facilitate the comparison of the problems treated in Sections VIII and IX with the computations that have been made for the tidal ebb and flow, I will allow myself to add some things that do not properly belong to the subject of this investigation. The following formulæ differ from the ordinary ones only in the notation, and in the fact that the velocities are retained in place of the displacements.*

In the rotating spherical shell of radius $S$, and of constant temperature $T$, the attraction of the sun produces motions for which the following equations, deduced from equations (7) and (10$a$), hold good:

$$\left.\begin{aligned} \frac{\partial (V-RT\varepsilon)}{S\,\partial\omega} &= \frac{\partial b}{\partial t} - 2\,n\,c\cos\omega \\ \frac{\partial (V-RT\varepsilon)}{S\sin\omega\,\partial\lambda} &= \frac{\partial c}{\partial t} + 2\,n\,b\cos\omega \\ \frac{\partial \varepsilon}{\partial t} + \frac{1}{S\sin\omega}\left(\frac{\partial (b\sin\omega)}{\partial\omega} + \frac{\partial c}{\partial\lambda}\right) &= 0 \end{aligned}\right\} \quad \ldots\ldots \quad 15.$$

$V$ indicates the potential of the sun at the point $(\omega, \lambda)$ of the rotating spherical shell. When the sun stands over the equator, its distance from the earth being $P$, its mass $M$, the constant of attraction $\varkappa$, we have then for the potential

$$\varkappa M\,[P^2 - 2\,P\,S\sin\omega\cos(nt+\lambda) + S^2]^{-\frac{1}{2}}$$

This being developed according to the powers of $\frac{S}{P}$ we obtain at first terms that have no, or at least very slight, import for the tidal ebb and flow; then come those that are to be subtracted when we consider the motion of the fluid as relative only to the center of gravity of the earth. That part of the potential which causes the semidiurnal tide we designate by $V$ in order to substitute it in the equations (15).

$$V = \frac{3}{4}\frac{\varkappa M S^2}{P^3}\sin^2\omega\cos(2\,nt + 2\,\lambda) = H(\omega)\cos(2\,nt + 2\,\lambda)$$

Put also

$$\begin{aligned} \varepsilon &= E(\omega)\cos(2\,nt + 2\,\lambda) \\ b &= \varphi(\omega)\sin(2\,nt + 2\,\lambda) \\ c &= \psi(\omega)\cos(2\,nt + 2\,\lambda) \end{aligned}$$

and

$$H - R\,T\cdot E = G(\omega)$$

and eliminate $\varphi$, $\psi$ from equations (15) we thus obtain

$$\frac{d^2 G}{d\omega^2}\sin^2\omega - \frac{dG}{d\omega}\sin\omega\cos\omega + G\,(4\,k\sin^4\omega + 2\sin^2\omega - 8) = 4\,k\,H\sin^4\omega \quad \ldots\ldots \quad (16)$$

* Compare, for example, the concise presentation by G. H. Darwin in the Encyclopædia Britannica, 9th edition, article "Tides."

This is the same as equation (13) of the previous section, only here $G$ replaces $E$, and $H$ replaces $A\ (\omega)$.

$$G=\frac{3}{4}\frac{\varkappa M S^2}{P^3}\ (\alpha_4 \sin^4 \omega+\alpha_6 \sin^6 \alpha+ \quad . \quad . \quad . \quad . \ )$$

$$E=\frac{1}{R\,T}\ \frac{3}{4}\frac{\varkappa M S^2}{P^3}\ (\sin^2 \omega-\alpha_4 \sin^4 \omega-\alpha_6 \sin^6 \omega- \quad . \quad . \quad . \quad . \ ) \quad . \quad . \quad (17)$$

For a given value of $T$ therefore, $\alpha_4$, $\alpha_6$, etc., are the same constants as in Section IX.

$m$ is the mass of the earth; $\frac{\varkappa m}{S^2}=g$; $M=355000\,m.$; $P=24000\ S$;

$$\frac{3}{4}\frac{\varkappa M S^2}{P^3}=1.203$$

Hence on the equator when $4\ k=10$, or $T=298.7$, we have

$$760\ \varepsilon=\frac{760}{287\times 298.7}\times 1.203\times 11.26\times \cos\,(2\,nt\times 2\lambda)$$

$$=0.12\ (mm)\ \cos\,(2\,nt+2\lambda)$$

Thus by the sun's attraction a semi-diurnal variation of the barometer of 0.24 *mm.* would arise at the equator; but through the moon's action one that is three times greater, 0.7 *mm.*

Laplace, in *Mécanique Céléste*, book IV, chapter 5, computed the atmospheric tide with the same value of $k$, but for an atmosphere over an ocean of constant depth, whose tides influence those of the air, whereas here the atmosphere over a rigid earth is alone considered. For our case the same formulæ obtain as for an ocean of uniform depth equal to $l$. In the equations (15) and subsequently, we have only to put $gl$ in place of $R\ T$, and $gy$ in place of $R\ T\ \varepsilon$, when $y$ is the elevation of the surface of the sea above the mean level.

The lunar tides computed from equations (17) with any allowable value of T are very much too large in comparison with those deduced from the barometer observations.* One can scarely wonder at this

* Besides the observations of Bouvard mentioned in Book XIII, *Mécanique Celéste* and which, arranged by syzygies and quadratures, show scarcely any difference in the daily variation of the barometer (note that only the observations of 9 A. M. and 3 P. M. were used), there are at hand for later dates computations of series of hourly observations for certain tropical stations that Professor Hann has pointed out to me. These give the following barometer variations produced by the lunar tides:

| | Latitude. | Altitude. | Barometric variation. | Authority. |
|---|---|---|---|---|
| | ° ′ | *metres.* | *mm.* | |
| Singapore ..... | 1 11 | ......... | 0.16 | Elliot, *Fortsch. d. Ph.*, 1852, p. 703. |
| Batavia......... | 6 11 | ......... | 0.115 | Bergsma, *Amsterdam Academy*, 1870. Vander Stok, *Batavia observations*, 1885, vol. 6. Sixteen accordant years. |
| St. Helena ...... | 15 57 | 540 | 0.10 | Sabine, *Fortsch. d. Ph.*, 1848, p. 402. |

The results for Singapore and St. Helena are remarkable in that the maxima occur precisely at the moment of lunar culmination; at Batavia the high tide is 50 minutes late.

when he reflects that all the hypotheses introduced into the computation (the neglect of the vertical motion, of the friction, and of the difference between a day and the interval between two lunar culminations) contribute to increase the computed tidal ebb and flow. With reference to the last-mentioned difference, I might further remark that it is easily introduced into the computation. The ratio 3 to 1 between the lunar tide and the solar tide as assumed by Laplace holds good only so long as the value $4k$ (in which the depth of the ocean or the temperature of the air enters in the respective problems) is far from a certain critical value which lies between 11.1 and 11.2. With $4k=10$ the ratio in question is 2.2 to 1, but with $4k=11.1$ the ratio becomes 1 to 5.

I do not carry out the computation here, because it seems too hypothetical to compare the atmosphere with a spherical shell of perfectly definite temperature, and under this assumption then to consider this semi-diurnal variation of pressure as a consequence of the solar attraction. Much more probable is it that it arises from a regular constituent of the semi-diurnal variation of temperature.

XX.

## LAPLACE'S SOLUTION OF THE TIDAL EQUATIONS.*

By William Ferrel.

In this paper (supplementary to that under the same heading in vol. IX, No. 6, of the *Astronomical Journal*), it is proposed to explain more fully a certain point in the latter (which did not appear clear to a correspondent some time since), by presenting the matter more in detail, and also to clear up some doubts held by some with regard to the convergency of the series in the tidal expression.

In Darwin's Equation No. (34),† we have the following differential equation to be satisfied, which is equivalent to that of Laplace:

$$\nu^2(1-\nu^2)\frac{d^2u}{d\nu^2} - \nu\frac{du}{d\nu} - (8-2\nu^2-\beta\nu^4)\,u+\beta\,E\nu^6=0 \quad . \quad . \quad . \quad . \quad (1)$$

[Darwin's Eq. (33).]

in which $u$ is the difference between the real amplitude of the tide and that given by the equilibrium theory, $\nu=\sin\vartheta$ is the sine of the geographical polar distance $\vartheta$, $E\nu^2$ is the amplitude of the equilibrium tide, and

$$\beta=\frac{4n^2}{gl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $\frac{n^2}{g}=\frac{1}{289}$ and $l$ is the depth of the ocean, supposed to be uniform, in terms of the earth's radius.

Putting

$$u=K_2\nu^2+K_4\nu^4+K_6\,\nu^6 \;\ldots\ldots\ldots\ldots\; K_n\nu^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $n$ is any even number, corresponding with the exponent, and substituting this value of $u$ and its derivatives in (1) above, we get, by equating the coefficients of like powers of $\nu$ to 0,

$$K_2=0. \quad 12K_4-12K_4=0. \quad 16K_6+\beta E=0, \text{ etc.,}$$

* From Gould's *Astronomical Journal*, 1890, vol. x, pp. 121–125.

† Encyclopedia Britannica, 9th ed. art. "Tides," § 16, vol. XXIII, p. 359.

$$u=(K^2-E)\,\nu^2+K_4\,\nu^4+K_6\,\nu^6+\,..\,K_{2i}\,\nu^{2i} \quad . \quad . \quad . \quad (34.)$$

and generally after $K_6$.

$$(n\,(n-2)-8)\,K_n-[(n-2)(n-3)-2]\,K_{n-2}+\beta K_{n-4}=0.$$

From these equations we get the following expressions of $K_n$:

$$\left.\begin{array}{l} K_2=0 \\ K_4=K_4 \\ K_6=\frac{5}{8}\,K_4-\frac{1}{16}\,\beta E \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$K_8=\left(\frac{7}{16}-\frac{1}{40}\beta\right)K_4-\frac{7}{160}\beta E.$$

$$\mathrm{K}_{10}=\left(\frac{21}{64}-\frac{79}{2880}\beta\right)K_4-\left(\frac{21}{640}-\frac{1}{1152}\beta\right)\beta \mathrm{E}..$$

and generally, after $K_6$,

$$K_n\,\frac{n-1}{n+2}K_{n-2}-\frac{\beta}{(n+2)\,(n-4)}\,K_{n-4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This general expression is equivalant to Laplace's and Darwin's law as given in my preceding paper, equation (2), but is more simple and convenient in deducing any coefficient $K_n$ from the last two preceding. The one is reducible to the other by putting $n=2i+4$. The general law of (5) does not hold until after $K_6$, but $K_4$ and $\mathrm{K}_6$ being obtained from the direct equation of the coefficients of $\nu^4$ and $\nu^6$, then by means of these, $K_8$ is obtained, either directly from the equation of the coefficients, or from the general expression of (5), and this law can be extended forward, but not backward. For instance, $K_6$ is not obtainable from $K_4$ and $K_2$. As is usual in such cases, the general law is not obtained until after several equations of the coefficients, and when the values of $K_n$ are given directly in this way, and not by the general law, the former must be taken, and the general law, which is a relation found between the coefficients after $K_6$ only, can not be extended back.

Putting $h$ for the amplitude of the real tide, we have, from what has been stated above,

$$h=E\nu^2+u=E\nu^2+K_4\nu^4+K_6\nu^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad \cdot \quad +K_n\nu^n \quad . \quad . \quad (6.)$$

Laplace extended the relation above, found to exist between the coefficients of $\nu$ in (3), and after $K_6$ only, back so as to make it, by means of the continued fraction, determine the value of $K_4$ and so the relation between $E\nu^2$ and $u$. This makes $K_4$ a determinate quantity, whereas the equation of the coefficients of $\nu^4$ gives $K_4=\mathrm{K}_4$, an indeterminate quantity. It is evident that any value of $\mathrm{K}_4$ satisfies the differential equation, and so, with the other coefficients depending upon it, is a solution of the tidal equation.

The extension of the general relation of (5) back so as to make it determine $\mathrm{K}_4$, and the relation between $E\nu^2$ and $u$ in (6), was regarded by the writer in his previous paper as an extension of the law back where

it is not applicable, and this is what was not clearly understood by his correspondent.

From (4) it is seen that the tidal expression consists of two parts, one of which depends upon $K_4$, and is independent of the tidal forces contained in $E$, and the latter depends upon these forces. It is evident that the former can exist without the latter. Also that being independent of the forces, and dependent simply upon certain initial motions which the sea may be supposed to have independent of the forces, it must vanish when there is friction, and so $K_4$ must be put equal to 0 in the real case of nature.

We come now to the second part of what we have proposed to consider here, namely, the convergency of the series in the expression of $u$ in (3). Inasmuch as the vanishing ratio between consecutive values of $K_n$ is unity, as is readily seen from an inspection of (5), it has been said that the device of Laplace in the use of the continued fraction was necessary to make the expression of $u$ convergent at the equator where $\nu = 1$, so as to give a finite value of $u$. It is true that the expression at first is more convergent with a large value of $K_4$, such as is given by the continued fraction, but still the vanishing ratio in any case is unity. But it can be shown that the expression gives a finite value of $u$ when we put $K_4 = 0$.

We get by development,

$$(1-\nu^2)^{\frac{1}{2}}=1-\frac{1}{2}\nu^2-\frac{1}{8}\nu^4-\frac{1}{16}\nu^6 \quad . \quad . \quad . \quad . \quad -A_n\nu^n=1+\Sigma_2^\infty A_n\nu_n \quad . \quad . \quad (7).$$

in which the relation between each coefficient $A_n$ and the preceding one, commencing with $-\frac{1}{2}$, is

$$A_n=\frac{n-3}{n}A_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence we have, when $\nu=1$

$$\Sigma_2^\infty A_n=-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

$$\Sigma_{n'+2}^\infty A_n=-\left(1+\Sigma_2^{n'} A_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

in which $n'$ is the exponent of any assumed term in the series.

The expression of (5) above may be put into the form,

$$K_n=\frac{n-3}{n}K_{n-2}+\frac{6}{n(n+2)}K_{n-2}-\frac{\beta}{(n+2)(n-4)}K_{n-4} \quad . \quad . \quad (11).$$

From this, by means of (8), we get for any coefficient for which the characteristic is $n'$,

$$K_{n'}=\frac{K_{n'-2}}{A_{n'-2}}A_{n'}F_{n'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

in which,

$$F_{n'} = \left(1 + \frac{6}{(n'+2)(n'-3)} - \frac{n'\beta}{(n'+2)(n'-4)(n'-3)}\right)\frac{K_{n'-4}}{K_{n'-2}} \quad . \quad . \quad (13)$$

and putting $n'+2$ for $n'$ in (12) we get

$$K_{n'+2} = \frac{K_{n'}}{A_{n'}} A_{n'+2} F_{n'+2}$$

$$K_{n'+4} = \frac{K_{n'+2}}{A_{n'+2}} A_{n'+4} F_{n'+4}.$$

This becomes by substituting for $\frac{K_{n'+2}}{A_{n'+2}}$ its value derived from the preceding expression becomes,

$$K_{n'+4} = \frac{K_{n'}}{A_{n'}} A_{n'+4} F_{n'+2} F_{n'+4}$$

In like manner we get generally

$$K_{n'+i} = \frac{K_{n'}}{A_{n'}} A_{n'+i} F_{n'+2} F_{n'+4} \quad . \quad . \quad . \quad . \quad . \quad F_{n'+i} \quad . \quad . \quad . \quad (14).$$

in which the values of the factors $F_{n'+2}$, $F_{n'+4}$, $F_{n'+i}$, are given by (13) by adding 2, 4, and $i$ respectively to $n'$ in that expression.

Now, all these factors are finite, and hence putting now $K_n$ for its equivalent, $K_{n'+i}$ and $A_n$ for $A_{n'+i}$, we have

$$\Sigma_{n'+2}^{\infty} K_n = \text{a finite quantity}$$

since by (10)

$$\Sigma_{n'+2}^{\infty} A_n = \text{a finite quantity}.$$

From (13) and (14) it is seen that any coefficient, taken without regard to signs,

$$K_{n'+i} < \frac{K_{n'}}{A_{n'}} A_{n'+i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

when

$$\beta > \frac{6\,(n'+i-4)}{n'+i} \; \frac{K_{n'+i-2}}{K_{n'+i-4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

since when this condition is satified all the factors $F_{n'+2}$, $F_{n'+4}$, . . . $F_{n'+i}$ are less than unity. Therefore, we have, putting $n$ for $n' + i$,

$$\Sigma_{n'+2}^{\infty} K_n \quad . \quad . \quad . \quad < \Sigma_{n'+2}^{\infty} \frac{K_{n'}}{A_{n'}} A_n \text{, or by (10),} < -\frac{K_{n'}}{A_{n'}}\left(1 + \Sigma_2^{n'} A_n\right) \qquad (17)$$

when

$$\beta > 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

since this is what (16) becomes when $i$ is infinitely great. This is simply the limiting condition in all cases, and the first number of (17) is generally less than the second when $\beta$ is considerably less than 6.

We have from (3)

$$u = \Sigma_2^{\infty} K_n = P_n + Q_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which

$$\left.\begin{aligned} P_n &= \Sigma_2^n K_n, \\ Q_n &= \Sigma_{n+2}^\infty K_n < -\frac{K_n}{A_n}(1-\Sigma_2^n A_n) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

With the values of $P_n$ and $Q_n$ (19) gives $u$, and this in (6) gives $h$, the amplitude of the tide.

Laplace computed the values of $2h$, that is, the range of the tides at the equator, at the times of conjunction of the moon and sun, for the several values of $\beta$ equal 40, 10, and 5, to which, by (2), correspond the several values of $l$, the depth of the ocean, equal to $\frac{1}{2833}$, $\frac{1}{708.2}$, and $\frac{1}{354.1}$ of the earth's radius, or approximately 1.4, 5.5, and 11 miles respectively.

Taking as an example the case in which $\beta=10$, we get from (4) and (5) by putting $K_4 = 0$, the following values of $K_n$ in terms of $E$ in the last column of the following table, and from (7) and (8) the corresponding values of $A_n$ in the second column.

| $n$ | $A_n$ | $K_n$ |
|---|---|---|
| 2 | −0.50000 | ........... |
| 4 | 0.12500 | ........... |
| 6 | .06250 | −1.4250 |
| 8 | .03906 | .4379 |
| 10 | .02734 | .2423 |
| 12 | .02051 | .1506 |
| 14 | .01611 | .1072 |
| 16 | .01309 | .0823 |
| 18 | .01091 | .0661 |
| 20 | .00927 | .0548 |
| . . | | |
| 40 | .00322 | .0172 |
| . . | | |
| 60 | .00174 | .0091 |

Putting $n$ equal 20, 40, 60, we get the following corresponding values from this table when complete for all the values of $n$ from 2 to 60,

$$\begin{aligned} A_{20} &= -.00927 \qquad 1+\Sigma_2^{20} A_n = .17621 \\ A_{40} &= -.00322 \qquad 1+\Sigma_2^{40} A_n = .12536 \\ A_{60} &= -.00174 \qquad 1+\Sigma_2^{60} A_n = .10254 \end{aligned}$$

From the values of $K_n$ we likewise get

$$\begin{aligned} K_{20} &= -.0548 \quad K_{20} / A_{20} = 5.91 \quad P_{20} = -1.7647 \\ K_{40} &= -.0172 \quad K_{40} / A_{40} = 5.18 \quad P_{40} = -2.0488 \\ K_{60} &= -.0091 \quad K_{60} / A_{60} = 5.23 \quad P_{60} = -2.1687 \end{aligned}$$

We therefore get from (19) and (20) with the preceding data,

$$\begin{aligned} u &< -1.7647 - 5.91 \times .17621 \text{ or } < -2.8061 \\ u &< -2.0488 - 5.18 \times .12536 \text{ or } < -2.7157 \\ u &< -2.1687 - 5.23 \times .10254 \text{ or } < -2.6870 \end{aligned}$$

and so on, according as we take $n'=20$, 40, 60, or still greater values. It is seen that the first value, in which we get the value of $P_{n'}$ from summing the actual values of $K_n$ from $n=6$ to $n=n'$, and then get the sum of the remaining infinite number of terms approximately from the last of (20), differs but little from the last value, in which the value of $P_{n'}$ was obtained from summing the actual values of $K_n$ up to $n'=60$, and then obtaining the sum of the remaining terms from the last of (20). It is evident that the real value of $u$ must be only a very little less negatively than $-2.6870$. The several values of $u$ differ the less, the more nearly the condition of (16) is satisfied, which, when the value of $n'$ is large, is very nearly that of (18). In our example $\beta=10$, and so is too large to give equal values in the several cases of $n'=20$, 40, or 60. With $\beta=40$ there is much greater difference in the several values, and the uncertainty in the last value is consequently much greater, but the last number so obtained is always a limit below which the real value is.

Since our values of $K_n$ have been computed in terms of $E$ the value of $u$ above must be multiplied into $E$. With this value, then, we get from (6) for the value of $h$ at the equator, where $\nu=1$,

$$h=(1-2.687)\,E=-1.687\,E.$$

The value of $E$ is that of the amplitude of the equilibrium tide at the equator, which in the case of the lunar tide, if we assume the moon's mass equal $\frac{1}{80}$, is 0.812 of a foot. Hence we get for the range of the lunar tide, approximately, at the equator,

$$2\,h=-2\times1.687\times0.812=-2.74 \text{ feet.}$$

Its being negative indicates that low water occurs at the time of the moon's meridian transit.

Laplace, in the same case, obtained for the range of the tide for the moon and sun in conjunction or opposition 11.05 metres, which, being positive, indicates that high water occurs at the time of meridian passage. But instead of $K_4=0$, he used $K_4=6.196$, obtained from his continued fraction. Besides, the mass of the moon which he used was much too large.